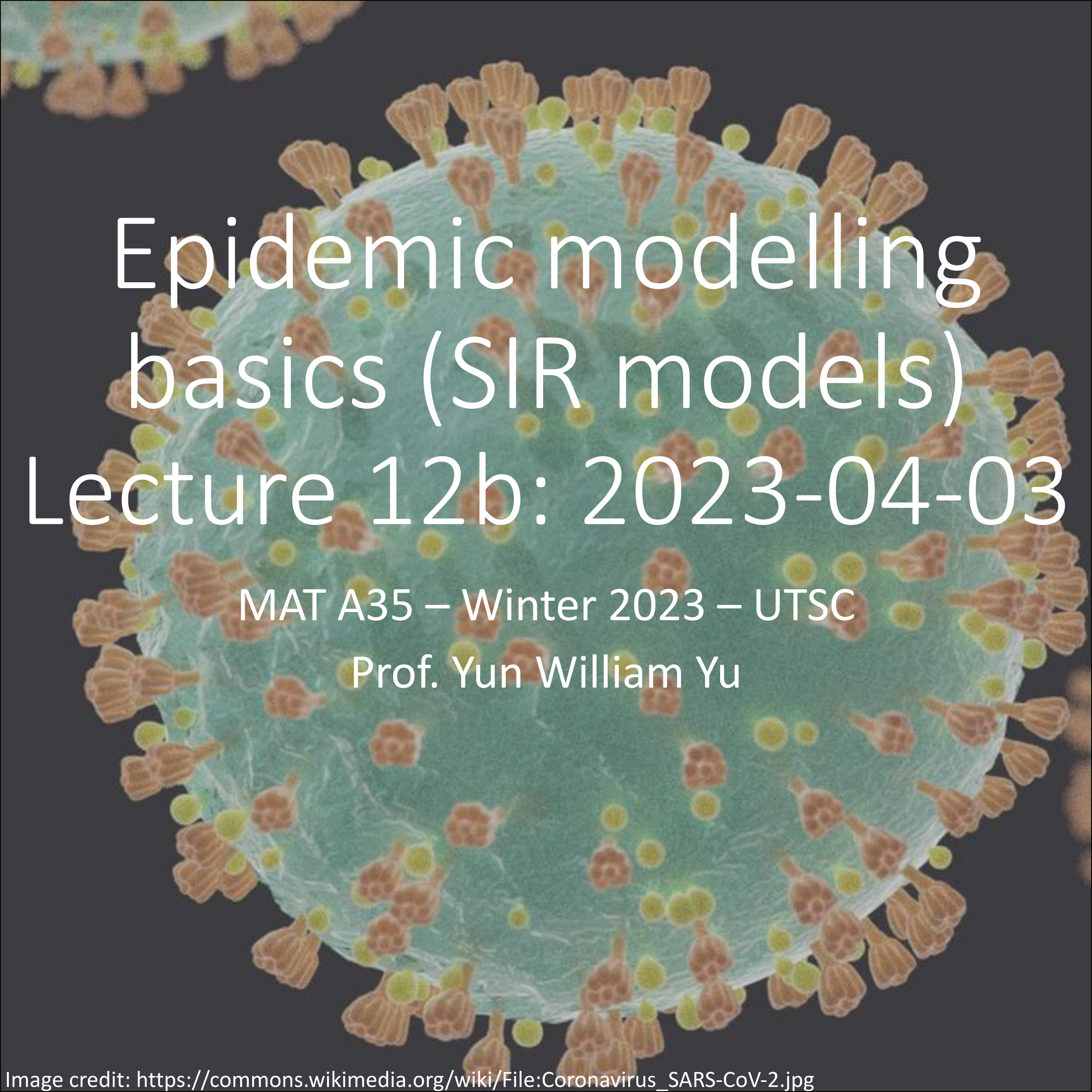


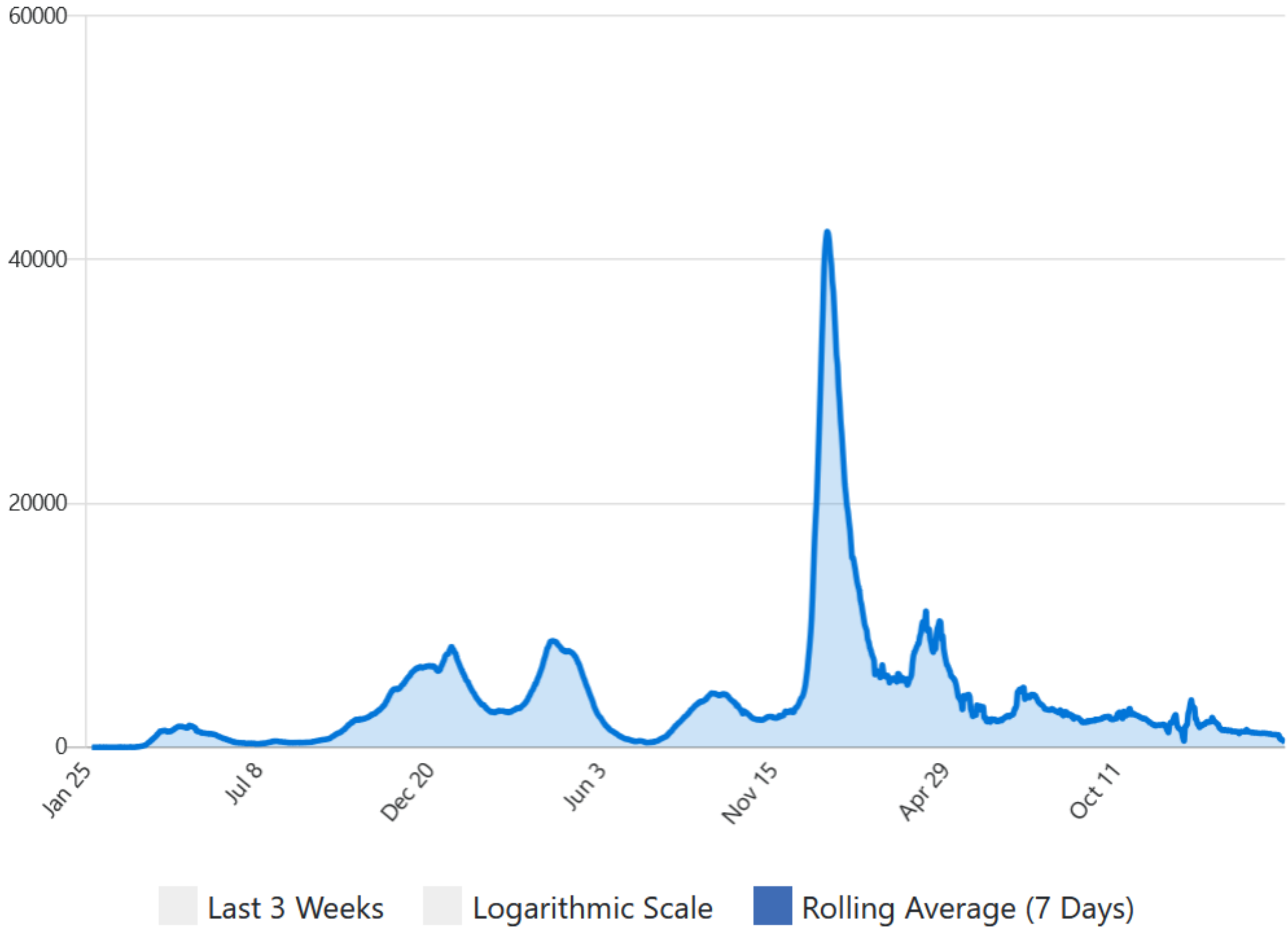
Epidemic modelling basics (SIR models) Lecture 12b: 2023-04-03

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

Image credit: https://commons.wikimedia.org/wiki/File:Coronavirus_SARS-CoV-2.jpg

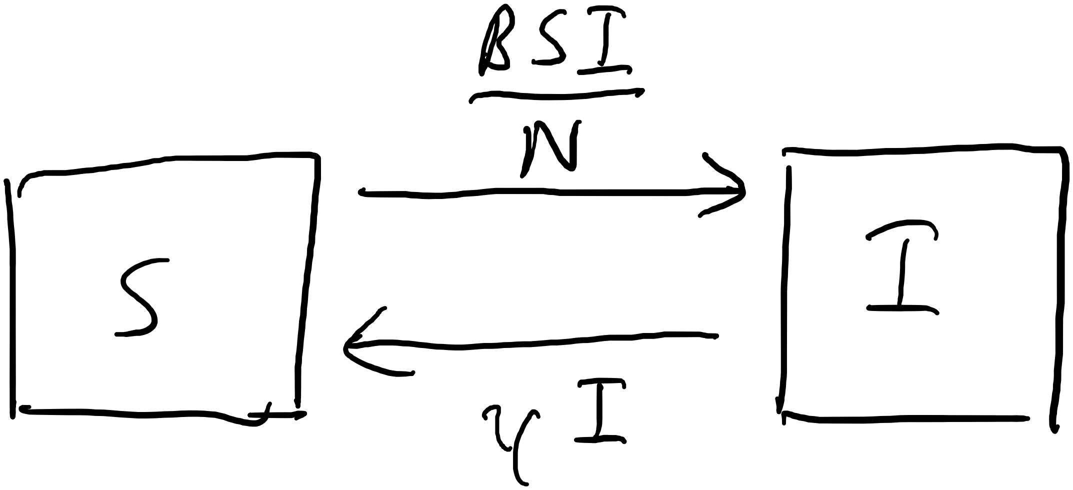
Epidemic curves – Covid19 in Canada



https://covid19tracker.ca/, 7-day rolling average, new cases

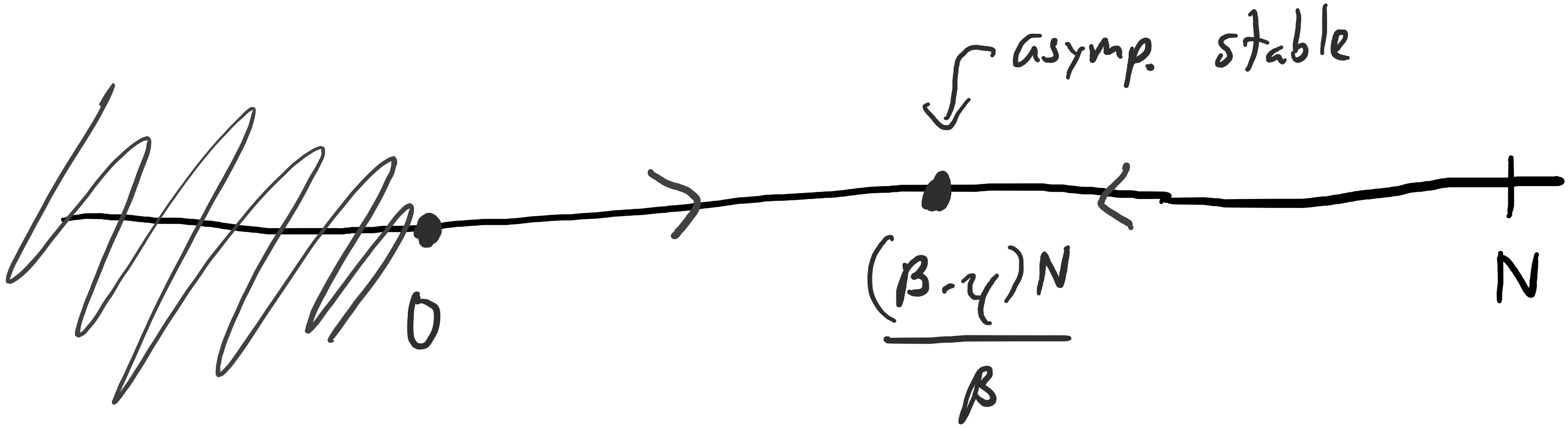
SIS Model

- Modified assumption 1: The infection rate is proportional to the average number of times an infected individual encounters a susceptible individual in the population, assuming random encounters.
- Assumption 2: there is a total fixed population size $N = I(t) + S(t)$, where S is the number of Susceptible individuals.
- Assumption 3: Individuals recover at rate γ, and become susceptible to re-infection.



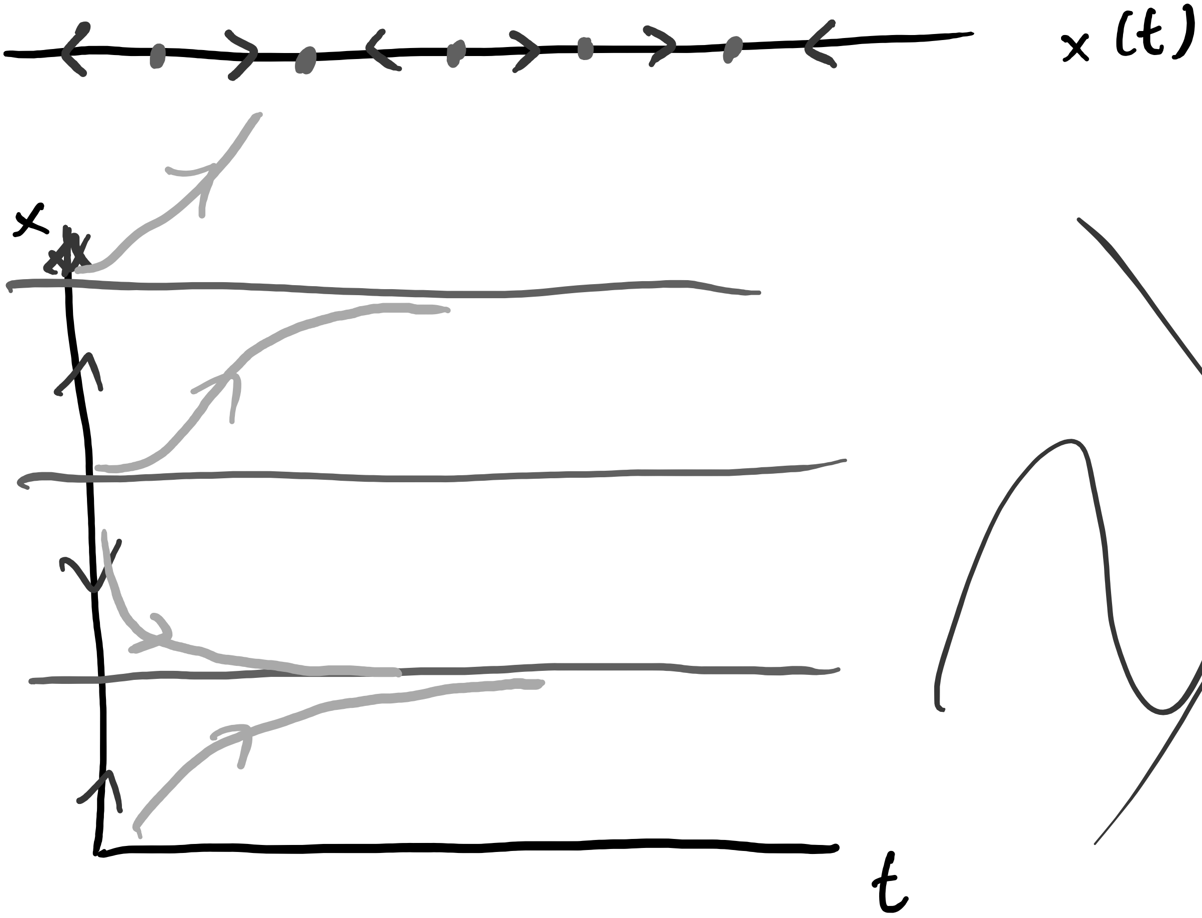
SIS Model Properties

- In the SIS model, we call the ratio $R_0 = \frac{\beta}{\gamma}$ the *basic reproduction number* of the system because it is the number of secondary infections β caused during the infectious period $\frac{1}{\gamma}$.
- If $R_0 > 1$, the disease persists and we are at the endemic disease equilibrium.



1D systems and complex behavior

- Autonomous 1D ODE systems $\dot{x} = f(x)$ can't have cycles or any doubling back because you'd have to cross a point going the other direction.



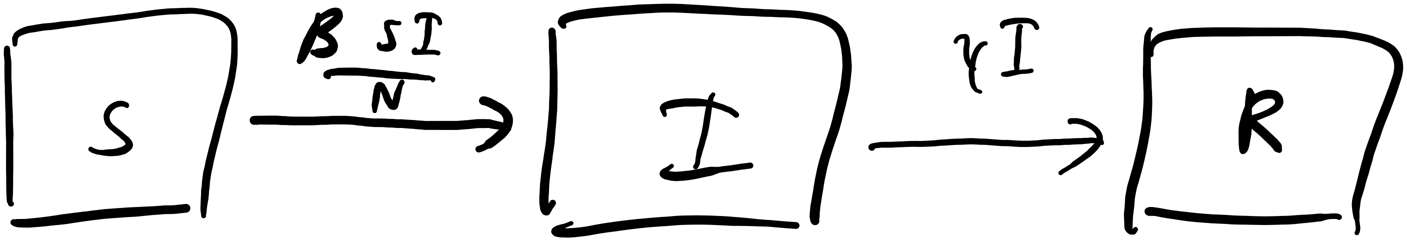
Modelling waves

- How can ODE models capture the infection waves that we see in real epidemics?

A: Add more variables
B: Make them nonautonomous
C: Add in births/deaths
D: All of the above
E: None of the above

SIR Model

- Modified assumption 3: Individuals recover at rate γ, and become *Removed* $R(t)$ from the population.
- Modified assumption 2: There is a total fixed population size $N = I(t) + S(t) + R(t)$.



$$\dot{S} = -\frac{\beta SI}{N}$$

$$\dot{I} = \frac{\beta}{N} SI - \gamma I \qquad \text{and} \qquad N = S + I + R$$

$$\dot{R} = \gamma I$$

Reduce SIR model to two variables

- $$\begin{cases} \dot{S} = -\frac{\beta}{N} SI \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

$N = S + I + R$

$\Rightarrow R = N - S - I$

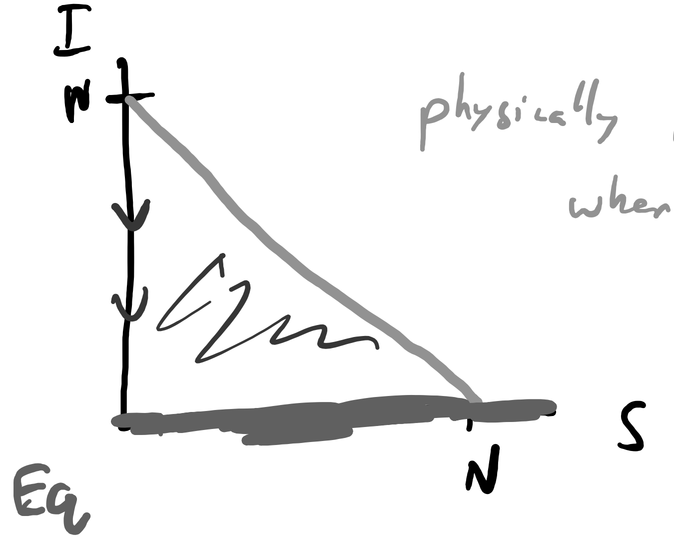
$$\begin{cases} \dot{S} = -\frac{\beta}{N} SI \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \end{cases}$$

Equilibria:

$$\left.\begin{aligned} \dot{S} = 0 &= -\frac{\beta}{N} SI \\ \dot{I} = 0 &= \frac{\beta}{N} SI - \gamma I \end{aligned}\right\} \Rightarrow -\gamma I = 0$$

$I = 0$

S can be anything

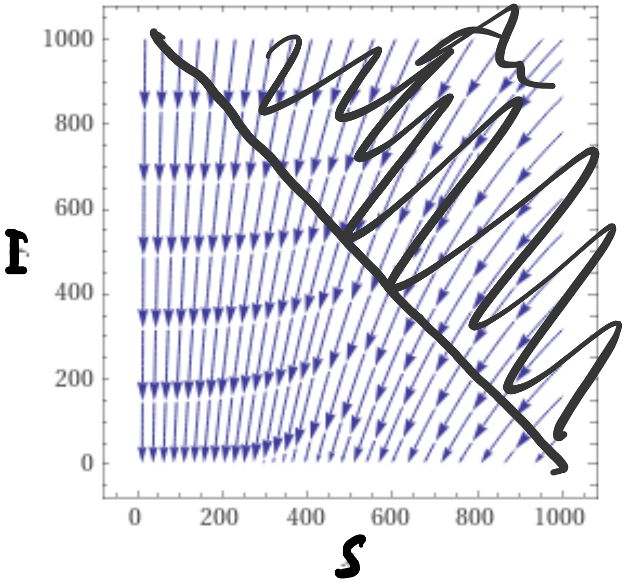


physically possible when $S + I \leq N$

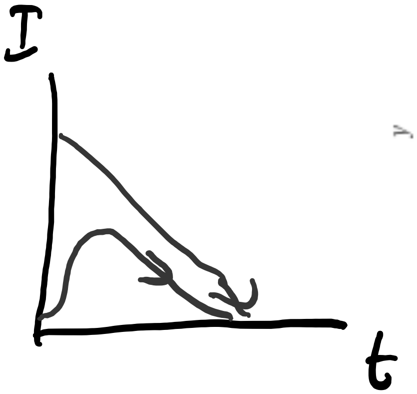
Plots of basic SIR model

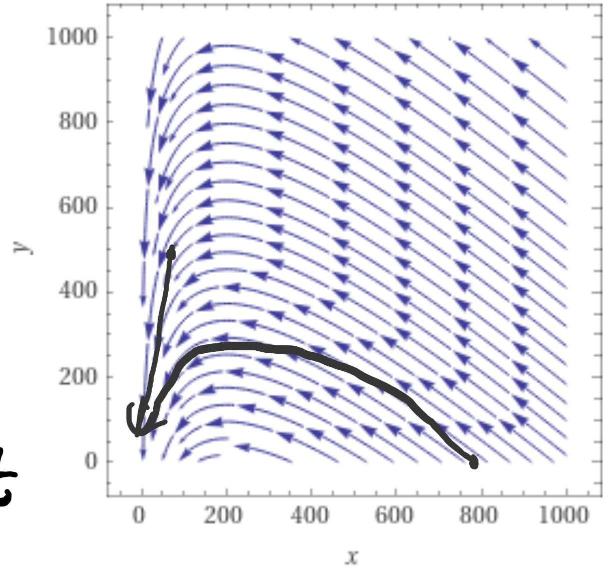
WolframAlpha: streamplot {-1/10000 * x * y, x*y/10000 - 0.2*y}, x=0..1000, y=0..1000

- Let $\beta = 0.1, \gamma = 0.2, N = 1000$
- $\begin{cases} \dot{S} = -\frac{0.1}{1000} SI \\ \dot{I} = \frac{0.1}{1000} SI - 0.2I \end{cases}$



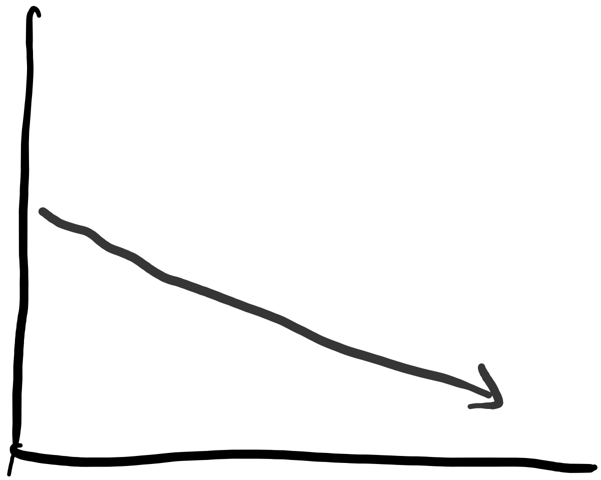
- Let $\beta = 0.1, \gamma = 0.02, N = 1000$
- $\begin{cases} \dot{S} = -\frac{0.1}{1000} SI \\ \dot{I} = \frac{0.1}{1000} SI - 0.02I \end{cases}$





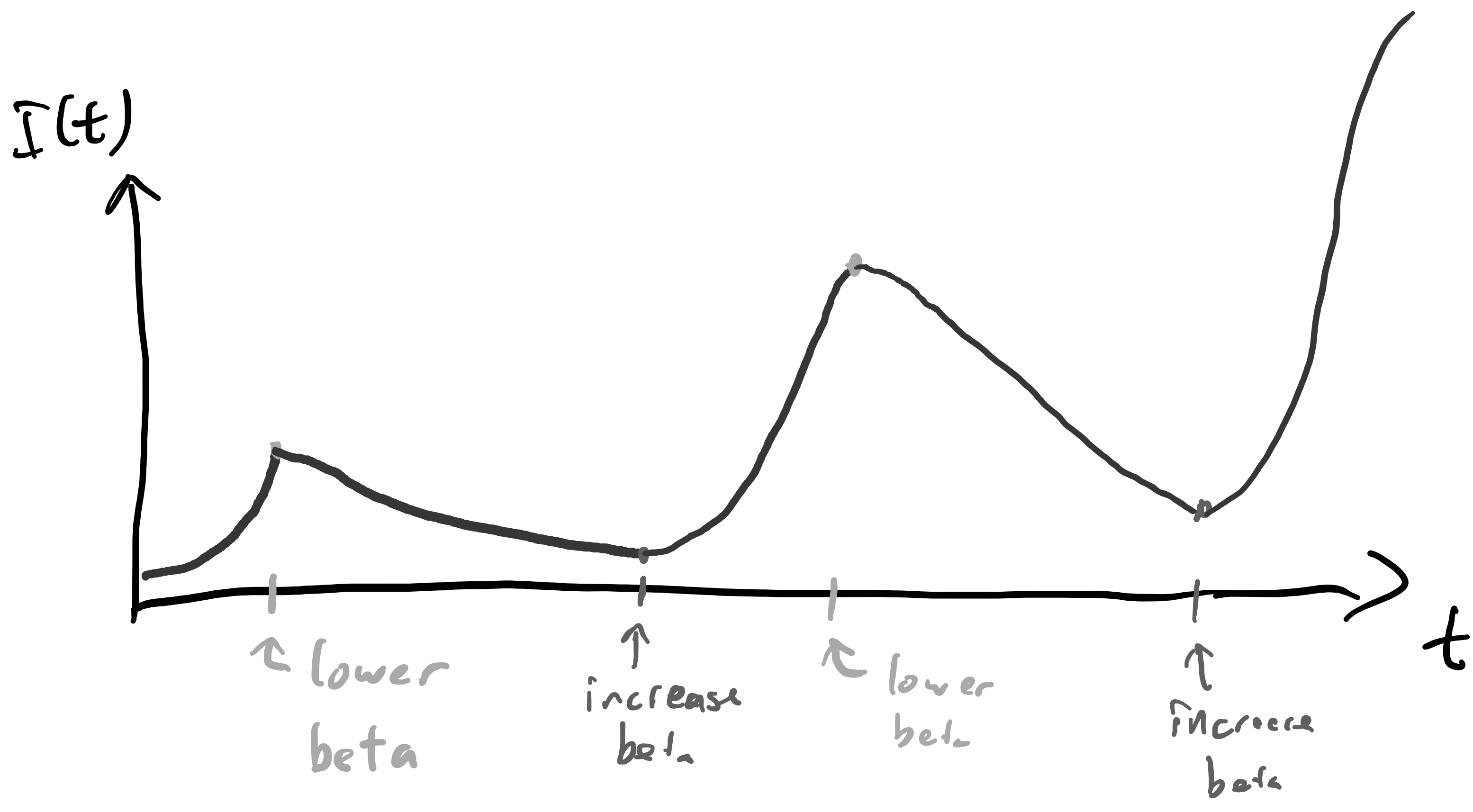
Basic SIR model gives a single wave

- Again, $R_0 = \frac{\beta}{\gamma}$.
- If $R_0 < 0$, then no epidemic happens because infected individuals never increase.
- If $R_0 > 1$, then an epidemic with a single wave happens.
- Then higher the R_0, the higher the peak. We can "flatten the curve" by reducing β, the transmission rate.
- https://www.geogebra.org/m/p5ucts38 by Ivan De Winne



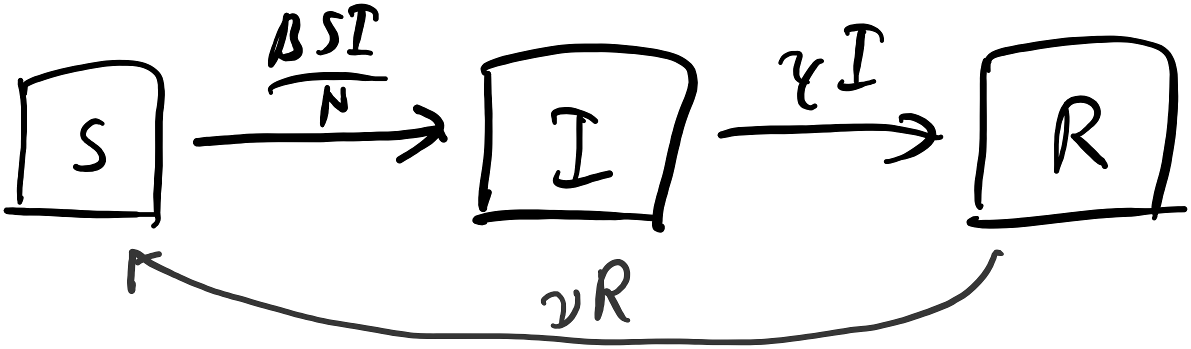
Nonautonomous Epidemic model

- What if β is sometimes high, and sometimes low, as a function of time?
- Then $\begin{cases} \dot{S} = -\frac{\beta(t)}{N} SI \\ \dot{I} = \frac{\beta(t)}{N} SI - \gamma I \end{cases}$ is not autonomous.



(autonomous) SIRS Epidemic Model

- Assumption 4: Removed individuals lose immunity at rate ν.



$$\begin{cases} \dot{S} = -\frac{\beta}{N} SI + \nu R \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \\ \dot{R} = \gamma I - \nu R \end{cases}$$

$$N = S + I + R$$

$$R = N - S - I$$

$$\begin{cases} \dot{S} = -\frac{\beta}{N} SI + \nu(N - S - I) \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \end{cases}$$

Phase plane analysis of SIRS model

- $$\begin{cases} \dot{S} = -\frac{\beta}{N}SI + \nu(N - I - S) \\ \dot{I} = \frac{\beta}{N}SI - \gamma I \end{cases}$$

$R_0 = \frac{\beta}{\gamma}$

Equilibria

- $0 = -\frac{\beta}{N}SI + \nu(N - I - S)$

$0 = \frac{\beta}{N}SI - \gamma I \Rightarrow 0 = I\left(\frac{\beta}{N}S - \gamma\right)$

$\Rightarrow I = 0 \quad \text{or} \quad S = \frac{N\gamma}{\beta}$

Case 1: $I = 0$

$\Rightarrow 0 = \nu(N - S)$

$\Rightarrow S = N$

Case 2: $S = \frac{N\gamma}{\beta}$

$0 = -\frac{\beta}{N} \cdot \frac{N}{\beta} \cdot \gamma \cdot I + \nu\left(N - I - \frac{N\gamma}{\beta}\right)$

$\Rightarrow \gamma I + \nu I = \nu N\left(1 - \frac{\gamma}{\beta}\right)$

$\Rightarrow I = \frac{\nu N\left(1 - \frac{\gamma}{\beta}\right)}{\gamma + \nu}$

Qualitative analysis of equilibria

- $$\begin{cases} \dot{S} = -\frac{\beta}{N} SI + \nu(N - I - S) \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \end{cases}$$

- Equilibria at

$$\begin{cases} (S, I) = (N, 0) \\ (S, I) = \left(\frac{N\gamma}{\beta}, \frac{\nu N\left(1-\frac{\gamma}{\beta}\right)}{\gamma+\nu}\right) \end{cases}$$

$$J(S,I) = \begin{bmatrix} -\frac{\beta I}{N} - \nu & -\frac{\beta S}{N} - \nu \\ \frac{\beta I}{N} & \frac{\beta S}{N} - \gamma \end{bmatrix}$$

$$J(N,0) = \begin{bmatrix} -\nu & -\beta - \gamma \\ \beta & \beta - \gamma \end{bmatrix}$$

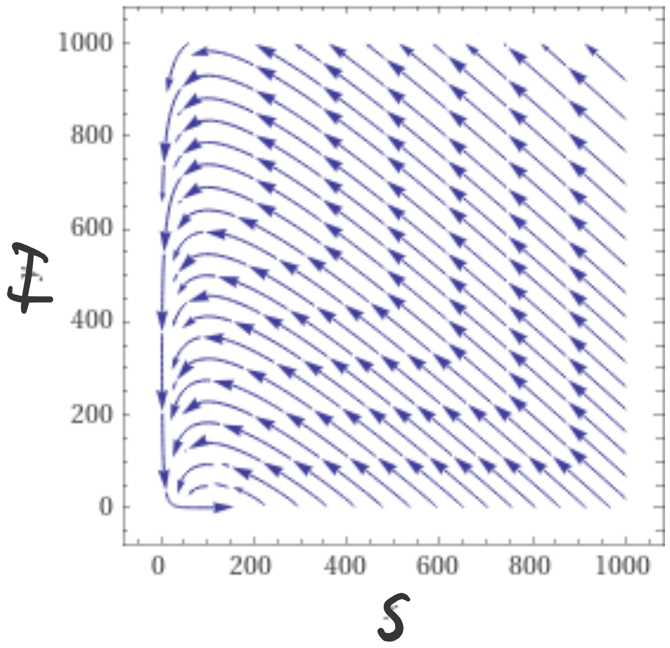
$$J(\cdots, \cdots) = \begin{bmatrix} -\frac{\nu\beta\left(1-\frac{\gamma}{\beta}\right)}{\gamma+\nu} - \nu & -\gamma - \nu \\ \frac{\nu\beta\left(1-\frac{\gamma}{\beta}\right)}{\gamma+\nu} & 0 \end{bmatrix}$$

Example with periodic behavior

- $\beta = 0.1, \gamma = 0.01, \nu = 0.0001, N = 1000$
- $$\dot{S} = -\frac{\beta}{N}SI + \nu(N - I - S)$$
  $$= -0.0001SI + 0.0001(1000 - S - I)$$
- $$\dot{I} = \frac{\beta}{N}SI - \gamma I = 0.0001SI - 0.01I$$
- Equilibria: $(S, I) = (1000,0)$ or $(S, I) \approx (100, 8.91)$
- Jacobian $J(S, I) = \begin{bmatrix} -0.0001I - 0.0001 & -0.0001S - 0.0001 \\ 0.0001I & 0.0001S - 0.01 \end{bmatrix}$
- $J(1000,0) = \begin{bmatrix} -0.0001 & -0.1001 \\ 0 & 0.09 \end{bmatrix}$,
  $\lambda_1 = -0.0001, \lambda_2 = 0.09$, saddle pt
- $J(100, 8.91) = \begin{bmatrix} -0.000991 & -0.0101 \\ 0.000891 & 0 \end{bmatrix}$,
  $\lambda_{1,2} \approx -0.0004955 \pm 0.00296i$ so the equilibrium is an attracting spiral

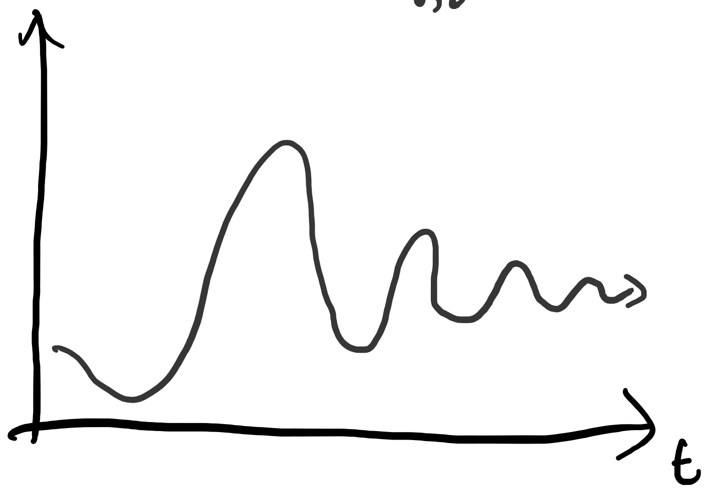
Visualization

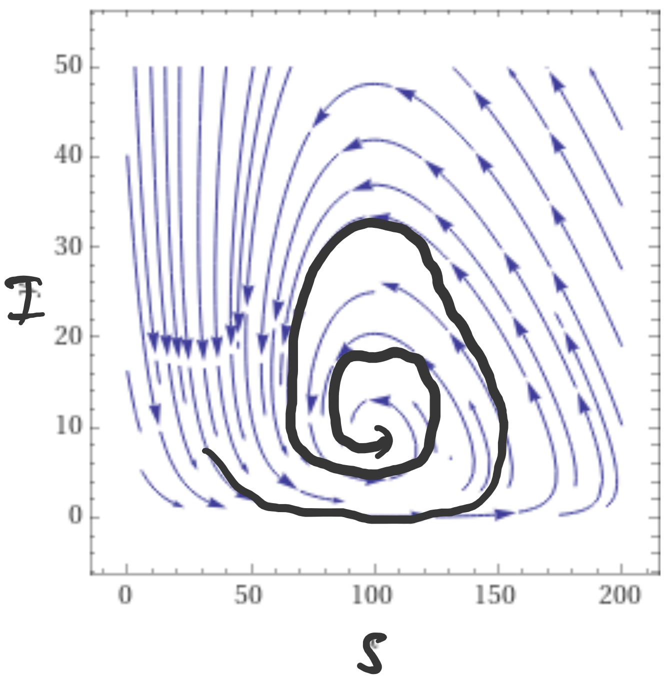
- streamplot {- x * y/10000 + 0.0001*(1000-x-y), x*y/10000 - 0.01*y}, x=0..1000, y=0..1000



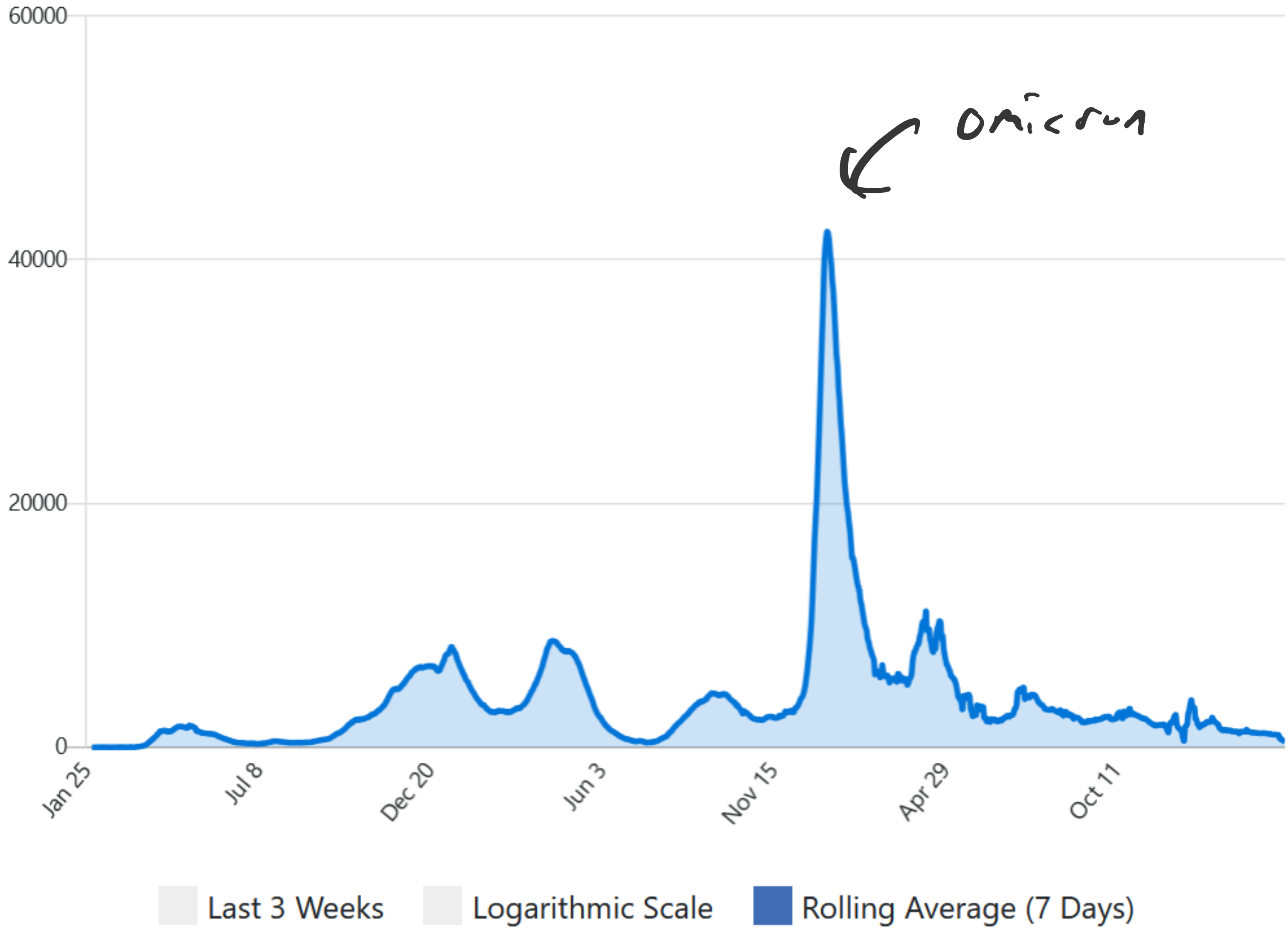
- streamplot {- x * y/10000 + 0.0001*(1000-x-y), x*y/10000 - 0.01*y}, x=0..400, y=0..50

$\lambda_{1,2} = -a \pm ib$





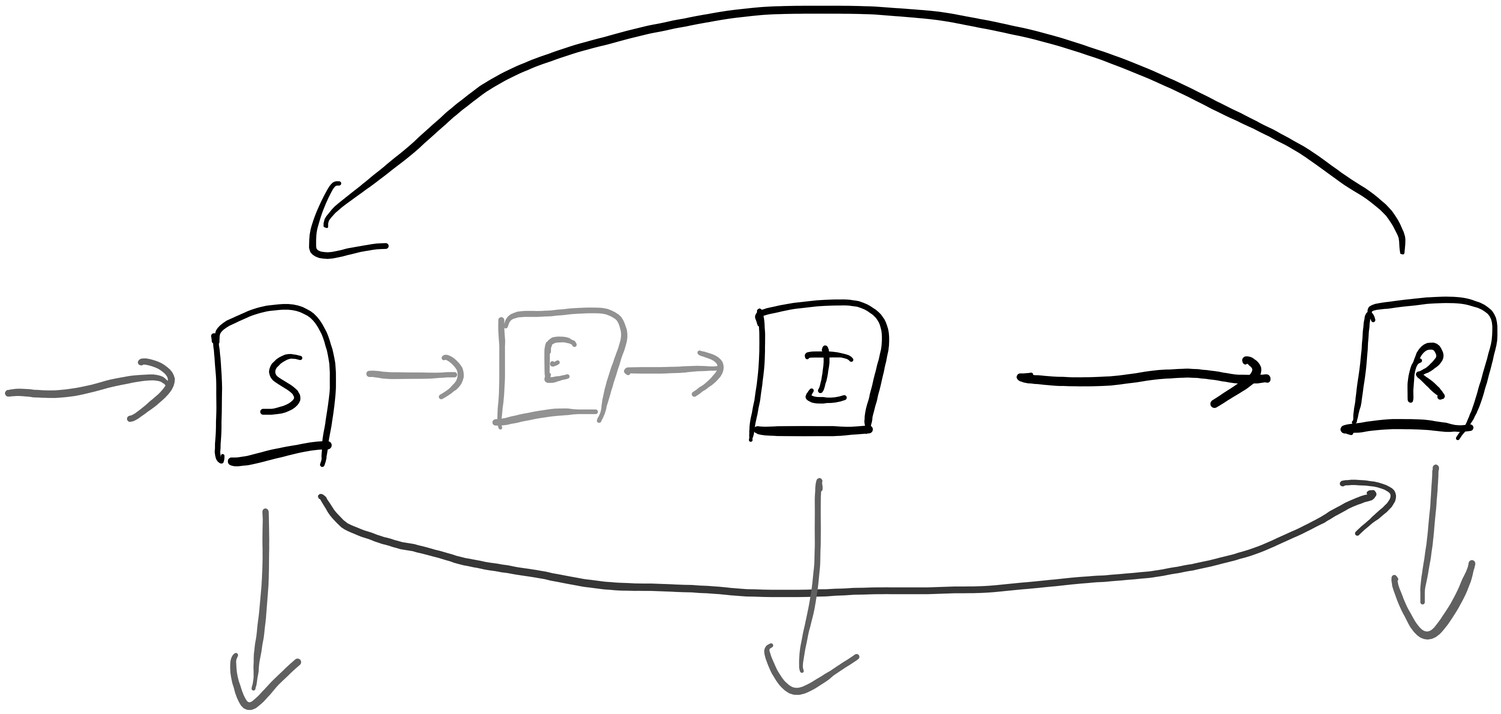
Epidemic curves – Covid19 in Canada



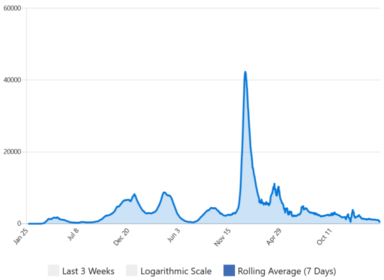
https://covid19tracker.ca/, 7-day rolling average, new cases

More sophisticated epidemic models

- Vaccinations = arrow directly from S → R
- Birth/Death = arrows entering S or leaving
- Exposed but not Infectious = new compartment



Modelling summary



- We can build models of complex biological phenomenon like epidemics by looking at behavior we care about and adding more assumptions to reproduce that real behavior in our models.
- Sometimes, we can get similar behavior with two different assumptions, so as mathematical biologists we need to decide what fits better.
- For epidemic modelling, these can include additional variables/compartments, as well as new arrows between compartments.