Final Review Session
Lecture 13: 2023-04-06

MAT A35 – Winter 2023 – UTSC

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Basic derivative/integration table

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\lvert x\rvert] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\;\,dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\;\,dx = \frac{1}{\mathrm{a}}\sin ax + C$

Derivative rules

$f(x) = x^2$ $\quad g(x) = 3x-1$

$f'(x) = 2x$ $\quad g'(x) = 3$

- Chain rule: $\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)g'(x)$

$$\frac{d}{dx}\left[(3x-1)^2\right] = 2(3x-1)\cdot 3$$

- Product rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

$$\frac{d}{dx}\left[x^2(3x-1)\right] = x^2\cdot 3 + 2x\cdot(3x-1)$$

- Quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$

$$\frac{d}{dx}\left[\frac{x^2}{3x-1}\right] = \frac{(3x-1)\cdot 2x - x^2\cdot 3}{(3x-1)^2}$$

Integration techniques

- Substitution method
  - Guess an appropriate u
  - Compute du, dx, and x
  - Substitute to get rid of x's
  - Integrate as a function of u
  - Convert back to x's
- Integration by parts
  - $\int u\,dv = uv - \boxed{\int v\,du}$
  - DETAIL heuristic to guess u vs. dv
  - Apply formula to see if it works.
- Partial fractions
  - $\frac{A}{ax+b} + \frac{B}{cx+d} = \frac{A(cx+d)+B(ax+b)}{(ax+b)(cx+d)}$

$$\int 2x e^{x^2} dx = \int e^u du = e^u + C = e^{x^2} + C$$

$u = x^2$

$du = 2x\,dx$

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

$u = x$ $\quad v = \frac{1}{2} e^{2x}$

$du = dx$ $\quad dv = e^{2x} dx$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$A(x-1) + B(x+1) = 1$$

$$(A+B)x + (B-A) = 1$$

$$\left.\begin{aligned} A+B &= 0 \\ B-A &= 1 \end{aligned}\right\} \quad A = -\frac{1}{2}, \quad B = \frac{1}{2}$$

$$\frac{1}{x^2-1} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}$$

Matrix multiplication

- Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix. Then the product $C = AB$ is a $m \times p$ matrix such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ik}b_{kj}$$

$$\underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 1\cdot 2 + 2\cdot 1 + 3\cdot 3 \\ 4\cdot 2 + 5\cdot 1 + 6\cdot 3 \\ 7\cdot 2 + 8\cdot 1 + 9\cdot 3 \end{bmatrix}} = \begin{bmatrix} 13 \\ 31 \\ 49 \end{bmatrix}$$

Matrix eigenvalues and eigenvectors

- A square matrix A's eigenpairs: (λ, v) such that $Av = \lambda v$.
- You can compute the eigenvalues by $\det(\lambda I - A) = 0$.
- Then you can compute the eigenvectors by solving.

$\det(A - \lambda I) = 0$

$$\begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 7 & -8 \\ 4 & \lambda + 5 \end{vmatrix} = \lambda^2 - 2\lambda - 35 + 32 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1, \quad 3$$

$\lambda_1 = -1$

$$\begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$7x + 8y = -x$$

$$8y = -8x$$

$$y = -x$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Leslie diagrams and matrices

- Leslie diagram arrows represent how each life stage gives rise to individuals in the next life stage.
- Leslie matrices encode that into a matrix; each column encodes all arrows that starts from the corresponding node. Each row encodes all arrows that end in the corresponding node.

$$
\begin{tikzcd}
H \arrow[r, "0.5"] & J \arrow[r, "0.9"] & A \arrow[ll, bend left, "5"]
\end{tikzcd}
$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} H & J & A \end{array} \\
\begin{array}{c} H \\ J \\ A \end{array} & \begin{bmatrix} 0 & 0 & 5 \\ 0.5 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix}
\end{array}
$$

Leslie matrices and population prediction

- If we are given a Leslie matrix L and a current population vector p, then the population one "cycle" later will be Lp, two cycles later will be $L \cdot Lp = L^2p$, etc.
- Furthermore, the population one cycle earlier can be computed by solving the equation $Lx = p$, or by using the matrix inverse and computing $x = L^{-1}p$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

Gauss-Jordan

Gaussian elim

$$\left[\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right]$$

$\underbrace{\begin{matrix} a & b \\ c & d \end{matrix}}_{\text{inverse}}$

Separation of variables

- Let $\frac{dy}{dx} = f(x)g(y)$.
- Then $\frac{dy}{g(y)} = f(x)dx$.
- Integrate both sides.

$$y' = 2x + xy$$

$$\frac{dy}{dx} = x(2+y)$$

$$\int \frac{dy}{2+y} = \int x \, dx$$

$$\ln|2+y| = \frac{1}{2}x^2 + C$$

$$y+2 = Ce^{0.5x^2}$$

$$y = -2 + Ce^{0.5x^2}$$

Exact differentials

- $P(x,y)dx + Q(x,y)dy = 0$, where there exists a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$— Alternate check: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- Then $f(x,y) = C$

$$(2xe^{x^2} + y)dx + (x)dy = 0$$

$$\int [2xe^{x^2} + y]dx = e^{x^2} + xy + F(y)$$

$$\int (x)dy = xy + G(x)$$

$$\Rightarrow e^{x^2} + xy + C = 0$$

Constant coefficient homogeneous

- Find all roots $\lambda_1, \dots, \lambda_n$ of characteristic polynomial.
- A root with multiplicity 1 means that $e^{\lambda x}$ is a solution.
- A root with multiplicity k means that $x^{k-1}e^{\lambda x}$ is a solution.
- Take all linear combinations of those solutions.

$y'' + 4y' + 4y = 0$

Char eq $\lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$

$\lambda = -2$, mult 2

Solutions $= e^{-2x}, \ xe^{-2x}$

$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

$y'' + 4y = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

Sol: $\cos 2x, \ \sin 2x$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

Method of undetermined coefficients

- Applicable to constant coefficient linear inhomogeneous ODEs.
- First find homogeneous solution.
- Then guess an Ansatz for the particular solution that has terms corresponding to each of the derivatives of the terms in the RHS.
- Get general solution by combining homogeneous and particular solutions.

Homogeneous linear systems

- Given a matrix ODE $\dot{z} = Az$, if there is an eigenbasis for A, then $z = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t}$, where (λ_i, v_i) are eigenpairs.

$\dot{x} = 7x + 8y$

$\dot{y} = -4x - 5y$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

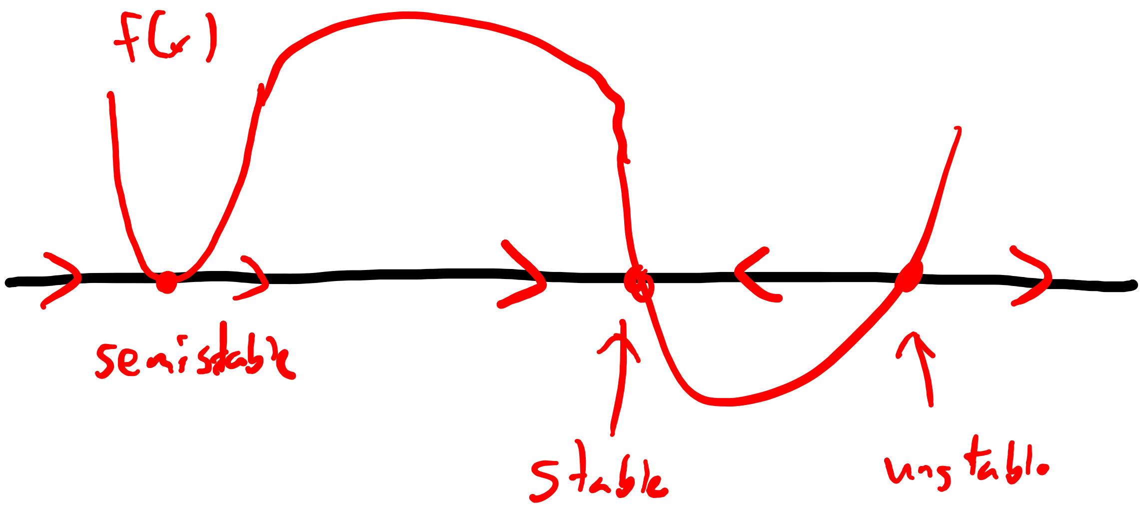
$\lambda_1 = -1$ $\quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$ $\quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Phase lines

- For a 1-variable autonomous ODE $\dot{x} = f(x)$, we can draw a phase line by looking at the sign of $\dot{x}$.
- Equilibria are at points where $\dot{x} = 0$.
- If $\dot{x} > 0$, then arrows point right-ward.
- If $\dot{x} < 0$, then arrows point left-ward.
- If both arrows point inward to an equilibrium, asymptotically stable.
- If both arrows point outward from an equilibrium, then unstable.
- If one points inward and the other outward, then semi-stable.



Critical points of multivariable function

- Given $f(x, y)$, the critical points are where $f_x = 0$ and $f_y = 0$.
- The Hessian matrix is $H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.
- If the Hessian matrix at a critical point has all positive eigenvalues, then the critical point is a local minimum.
- If the Hessian matrix at a critical point has all negative eigenvalues, then the critical point is a local maximum.
- If the Hessian matrix has opposite-sign critical points, then it is a saddle point.

Stability analysis: autonomous 2D system

- Consider a linear autonomous system $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$.
- Equilibria are where $\dot{x} = 0$ and $\dot{y} = 0$.
- The Jacobian matrix is $\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$, and its eigenvalues at an equilibrium determine its classification/stability.
- Positive real parts mean that trajectories go outward.
- Negative real parts mean that trajectories go inward.
- Opposite sign eigenvalues mean you have a saddle point.
- Nonzero imaginary components mean that trajectories spiral.

Classification of types

- Nodes: both eigenvalues are real and have the same sign. Unstable node if both positive, asymptotically stable node if both negative.
- Saddle point: both eigenvalues are real and have opposite sign.
- Spirals: complex eigenpair. If real parts are positive, unstable. If real parts are negative, asymptotically stable.
- Center: pure imaginary eigenpair. “stable”

Power series

- $f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$
- Also, power series can be manipulated like polynomials.
  - This includes, addition, subtraction, multiplication, and derivatives.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \frac{d}{dx}[\sin x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(x+1) = 1 - \frac{(x+1)^2}{2!} + \frac{(x+1)^4}{4!} - \frac{(x+1)^6}{6!} + \cdots$$