Final Review Session
Lecture 13: 2023-04-06

MAT A35 – Winter 2023 – UTSC

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Basic derivative/integration table

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\,dx = \ln\|x\| + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\;\,dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\;\,dx = \frac{1}{\mathrm{a}}\sin ax + C$

Derivative rules

- Chain rule: $\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)g'(x)$

- Product rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

- Quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Integration techniques

- Substitution method
  - Guess an appropriate u
  - Compute du, dx, and x
  - Substitute to get rid of x's
  - Integrate as a function of u
  - Convert back to x's
- Integration by parts
  - $\int u\,dv = uv - \int v\,du$
  - DETAIL heuristic to guess u vs. dv
  - Apply formula to see if it works.
- Partial fractions
  - $\frac{A}{ax+b} + \frac{B}{cx+d} = \frac{A(cx+d)+B(ax+b)}{(ax+b)(cx+d)}$

Matrix multiplication

- Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix. Then the product $C = AB$ is a $m \times p$ matrix such that
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ik}b_{kj}$$

Matrix eigenvalues and eigenvectors

- A square matrix A's eigenpairs: (λ, v) such that $Av = \lambda v$.
- You can compute the eigenvalues by $\det(\lambda I - A) = 0$.
- Then you can compute the eigenvectors by solving.

Leslie diagrams and matrices

- Leslie diagram arrows represent how each life stage gives rise to individuals in the next life stage.
- Leslie matrices encode that into a matrix; each column encodes all arrows that starts from the corresponding node. Each row encodes all arrows that end in the corresponding node.

Leslie matrices and population prediction

- If we are given a Leslie matrix L and a current population vector p, then the population one “cycle” later will be Lp, two cycles later will be $L \cdot Lp = L^2p$, etc.
- Furthermore, the population one cycle earlier can be computed by solving the equation $Lx = p$, or by using the matrix inverse and computing $x = L^{-1}p$.

Separation of variables

- Let $\frac{dy}{dx} = f(x)g(y)$.
- Then $\frac{dy}{g(y)} = f(x)dx$.
- Integrate both sides.

Exact differentials

- $P(x,y)dx + Q(x,y)dy = 0$, where there exists a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$— Alternate check: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- Then $f(x,y) = C$

Constant coefficient homogeneous

- Find all roots $\lambda_1, \ldots, \lambda_n$ of characteristic polynomial.
- A root with multiplicity 1 means that $e^{\lambda x}$ is a solution.
- A root with multiplicity k means that $x^{k-1}e^{\lambda x}$ is a solution.
- Take all linear combinations of those solutions.

Method of undetermined coefficients

- Applicable to constant coefficient linear inhomogeneous ODEs.
- First find homogeneous solution.
- Then guess an Ansatz for the particular solution that has terms corresponding to each of the derivatives of the terms in the RHS.
- Get general solution by combining homogeneous and particular solutions.

Homogeneous linear systems

- Given a matrix ODE $z = Az$, if there is an eigenbasis for A, then $z = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t}$, where (λ_i, v_i) are eigenpairs.

Phase lines

- For a 1-variable autonomous ODE $\dot{x} = f(x)$, we can draw a phase line by looking at the sign of $\dot{x}$.
- Equilibria are at points where $\dot{x} = 0$.
- If $\dot{x} > 0$, then arrows point right-ward.
- If $\dot{x} < 0$, then arrows point left-ward.
- If both arrows point inward to an equilibrium, asymptotically stable.
- If both arrows point outward from an equilibrium, then unstable.
- If one points inward and the other outward, then semi-stable.

Critical points of multivariable function

- Given $f(x, y)$, the critical points are where $f_x = 0$ and $f_y = 0$.
- The Hessian matrix is $H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.
- If the Hessian matrix at a critical point has all positive eigenvalues, then the critical point is a local minimum.
- If the Hessian matrix at a critical point has all negative eigenvalues, then the critical point is a local maximum.
- If the Hessian matrix has opposite-sign critical points, then it is a saddle point.

Stability analysis: autonomous 2D system

- Consider a linear autonomous system $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$.
- Equilibria are where $\dot{x} = 0$ and $\dot{y} = 0$.
- The Jacobian matrix is $\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$, and its eigenvalues at an equilibrium determine its classification/stability.
- Positive real parts mean that trajectories go outward.
- Negative real parts mean that trajectories go inward.
- Opposite sign eigenvalues mean you have a saddle point.
- Nonzero imaginary components mean that trajectories spiral.

Classification of types

- Nodes: both eigenvalues are real and have the same sign. Unstable node if both positive, asymptotically stable node if both negative.
- Saddle point: both eigenvalues are real and have opposite sign.
- Spirals: complex eigenpair. If real parts are positive, unstable. If real parts are negative, asymptotically stable.
- Center: pure imaginary eigenpair. “stable”

Power series

- $f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$
- Also, power series can be manipulated like polynomials.
  - This includes, addition, subtraction, multiplication, and derivatives.