Integration and antiderivatives Lecture 1a: 2023-01-09

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

Survey of backgrounds

- Type in chat "a", "b", "c", "d", or "e" to state your field of interest:
  - A: Biology
  - B: Chemistry
  - C: Mathematics
  - D: Environmental Science
  - E: Other

yuyun2 _ Prof Yu

- You may also add "?" to pump up the confusion meter to test out that functionality (or if you're confused).
- The "!" corresponds to your excitement level.

Re-invention of the trapezoid rule

A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves

MARY M. TAI, MS, EDD

OBJECTIVE — To develop a mathematical model for the determination of total areas under curves from various metabolic studies.

RESEARCH DESIGN AND METHODS — In Tai's Model, the total area under a curve is computed by dividing the area under the curve between two designated values on the *X*-axis (abscissas) into small segments (rectangles and triangles) whose areas can be accurately calculated from their respective geometrical formulas. The total sum of these individual areas thus represents the total area under the curve. Validity of the model is established by comparing total areas obtained from this model to these same areas obtained from graphic method (less than ±0.4%). Other formulas widely applied by researchers under- or overestimated total area under a metabolic curve by a great margin.

RESULTS — Tai's model proves to be able to *1*) determine total area under a curve with precision; *2*) calculate area with varied shapes that may or may not intercept on one or both *X/Y* axes; *3*) estimate total area under a curve plotted against varied time intervals (abscissas), whereas other formulas only allow the same time interval; and *4*) compare total areas of metabolic curves produced by different studies.

CONCLUSIONS — The Tai model allows flexibility in experimental conditions, which means, in the case of the glucose-response curve, samples can be taken with differing time intervals and total area under the curve can still be determined with precision.

Estimation of total areas under curves of metabolic studies has become an ... under a glucose-tolerance or an energy-expenditure curve (1,2) ...

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How long ago was the trapezoid rule invented?

A: Before 1 AD

B: 1-1000 AD

C: 1000-1500 AD

D: 1500-1900 AD

E: 1900-2000 AD

HISTORY OF SCIENCE

Ancient Babylonian astronomers calculated Jupiter's position from the area under a time-velocity graph

Mathieu Ossendrijver*

The idea of computing a body's displacement as an area in time-velocity space is usually traced back to 14th-century Europe. I show that in four ancient Babylonian cuneiform tablets, Jupiter's displacement along the ecliptic is computed as the area of a trapezoidal figure obtained by drawing its daily displacement against time. This interpretation is prompted by a newly discovered tablet on which the same computation is presented in an equivalent arithmetical formulation. The tablets date from 350 to 50 BCE. The trapezoid procedures offer the first evidence for the use of geometrical methods in Babylonian mathematical astronomy, which was thus far viewed as operating exclusively with arithmetical concepts.

The so-called trapezoid procedures examined in this paper have long puzzled historians of Babylonian astronomy. They belong to the corpus of Babylonian mathematical astronomy, which comprises about 450 tablets from Babylon and Uruk dating between 400 and 50 BCE. Approximately 340 of these tablets are tables with computed planetary or lunar data arranged in rows and columns (*1*). The remaining 110 tablets are procedure texts with computational instructions (*2*), mostly aimed at computing or verifying the tables. In all of these texts the zodiac, invented in Babylonia near the end of the fifth century BCE (*3*), is used as a coordinate system for computing celestial positions. The underlying algorithms are structured as branching ... of arithmetical operations (additions, sub... ...ns, and multiplications) that can be rep... ...ed as flow charts (*2*). Geometrical concepts ...nspicuously absent from these texts, whereas ...re very common in the Babylonian mathe... ...l corpus (*4*–*7*). Currently four tablets, most ...written in Babylon between 350 and 50 BCE, ...own to preserve portions of a trapezoid ...lure (*8*). Of the four procedures, here labeled ... (figs. S1 to S4), one (B) preserves a men... ...f Jupiter and three (B, C, E) are embedded in compendia of procedures dealing exclusively with Jupiter. The previously unpublished text D probably belongs to a similar compendium for Jupiter. In spite of these indications of a connection with Jupiter, their astronomical significance was previously not acknowledged or understood (*1*, *2*, *6*).

A recently discovered tablet containing an unpublished procedure text, here labeled text A (Fig. 1), sheds new light on the trapezoid procedures. Text A most likely originates from the same period and location (Babylon) as texts B to E (*8*). It contains a nearly complete set of instructions for Jupiter's motion along the ecliptic in accordance with the so-called scheme $X.S_1$ (*2*). Before the discovery of text A, this scheme was too fragmentarily known for identifying its connection with the trapezoid procedures. Covering one complete synodic cycle, scheme $X.S_1$ begins with Jupiter's heliacal rising (first visible rising at dawn), continuing with its first station (beginning of apparent retrograde motion), acronychal rising (last visible rising at dusk), second station (end of retrograde motion), and heliacal setting (last visible setting at dusk) (*2*). Scheme $X.S_1$ and the four trapezoid procedures are here shown to contain or imply mathematically equivalent descriptions of Jupiter's motion during the first 60 days after its first appearance. Whereas scheme $X.S_1$ employs a purely arithmetical terminology, the trapezoid procedures operate with geometrical entities.

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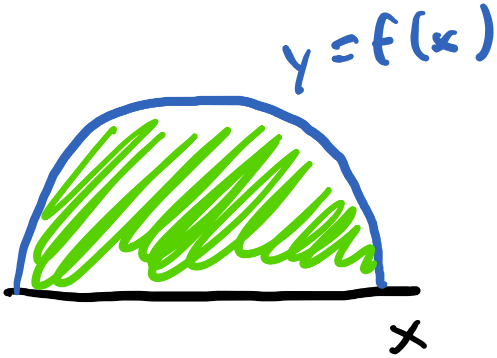
Differentiation and dimensional analysis

- Gets instantaneous rate of change of one variable vs another.
  - Let the variable x represent distance along the x-axis (in meters) and $y = f(x)$ be the height of the function on the y-axis (in meters).
    - Then $\frac{df}{dx} = f'(x)$ is the instantaneous slope (meters / meters = unitless).
  - Let the variable t represent time (in sec.) and the variable $x = x(t)$ be distanced traveled along the x-axis (in meters).
    - Then $\frac{dx}{dt} = \dot{x}(t)$ is the velocity (in meters/second)
  - Let the $v(t) = \dot{x}(t)$ be the velocity (meters/second, m/s)
    - Then $\frac{dv}{dt} = \ddot{x}(t)$ is the acceleration (in m/s^2)
  - Let $P(t)$ be the population (unitless count) at time t.
    - Then $\frac{dP}{dt}$ is the population growth rate (in number / second)

Integration

- "Opposite" of differentiation
  - "Summing up value as another variable changes"
  - Dimensional analysis: multiply together units along axes

Area:

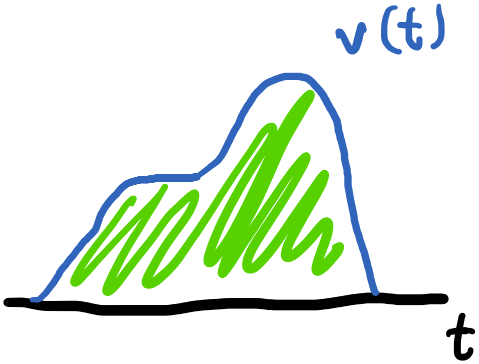


y-axis (meters)

x-axis (meters)

Area: m^2

Distance traveled



velocity (m/s)

time (s)

Distance (m)

Antiderivatives

$$f(x) = 35x + 2023 \xrightarrow{\text{differentiate}} f'(x) = 35$$

$$f'(x) = 35 \xrightarrow[?]{\text{antiderivative}} \int f'(x)\,dx = \int 35\,dx \overset{?}{=} 35x + 2023$$

But

$$\left.\begin{aligned} f(x) &= 35x + \pi \\ f(x) &= 35x - \sqrt{2} \\ f(x) &= 35x + 5 \end{aligned}\right\} f'(x) = 35 \rightarrow \int 35\,dx = 35x + C$$

Theorem: If two functions F and G have the same derivative $F'(x) = G'(x)$, then $F(x) = G(x) + C$, where C is a constant.

Can reverse many differentiation rules

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\,dx = \ln\|x\| + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\ dx = \frac{1}{\mathrm{a}}\sin ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\tan ax\right] = \sec^2 ax$	$\int \sec^2 ax\ dx = \frac{1}{\mathrm{a}}\tan ax + C$
$\frac{d}{dx}\left[-\frac{1}{\mathrm{a}}\cot ax\right] = \csc^2 ax$	$\int \csc^2 ax\ dx = -\frac{1}{\mathrm{a}}\cot ax + C$
$\frac{d}{dx}\left[\frac{1}{a}\sec ax\right] = \sec ax \tan ax$	$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec x + C$
$\frac{d}{dx}\left[-\frac{1}{a}\csc ax\right] = \csc ax \cot ax$	$\int \csc ax \cot ax\,dx = -\frac{1}{\mathrm{a}}\csc ax + C$

memorize

Advanced rules

- Constant multiplication rule

$$\int kf(x)dx = k\int f(x)dx$$

- Addition/subtraction rule

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$
$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

- No product or quotient rule. See: Integration by Parts.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

$$\int f(x)g(x)dx = ?$$

Examples

$\frac{d}{dx}\left[\frac{1}{3}x^3+C\right]=x^2$

$\int x^2\,dx = \frac{1}{3}x^3+C$

$\int 4\sin x\,dx = 4\int \sin x\,dx = -4\cos x + C$

$\int \frac{2}{a}\,da = 2\ln|a| + C$

$\int 5\,dy = 5y + C$

$\int (ma + a^{35})da = \int ma\,da + \int a^{35}\,da$

$= m\int a\,da + \int a^{35}\,da$

$= \frac{m}{2}a^2 + C_1 + \frac{1}{36}a^{36} + C_2$

$= \frac{m}{2}a^2 + \frac{1}{36}a^{36} + C$

$\int (ma + a^{35})da$, where m is a constant.

A: $m + 35a^{34}$

B: $m + 35a^{34} + C$

C: $\frac{m}{2}a^2 + \frac{1}{36}a^{36}$

D: $\frac{m}{2}a^2 + \frac{1}{36}a^{36} + C$

E: None of the above

Examples

$$\int \left[\cos 2x + 5e^{\pi x}\right] dx = \int \cos 2x \, dx + \int 5e^{\pi x} \, dx \qquad \text{(add rule)}$$

$$= \int \cos 2x \, dx + 5\int e^{\pi x} \, dx \qquad \left(\text{multiplication by constant}\right)$$

$$= \left[\frac{1}{2}\sin 2x + C_1\right] + 5\left[\frac{1}{\pi}e^{\pi x} + C_2\right]$$

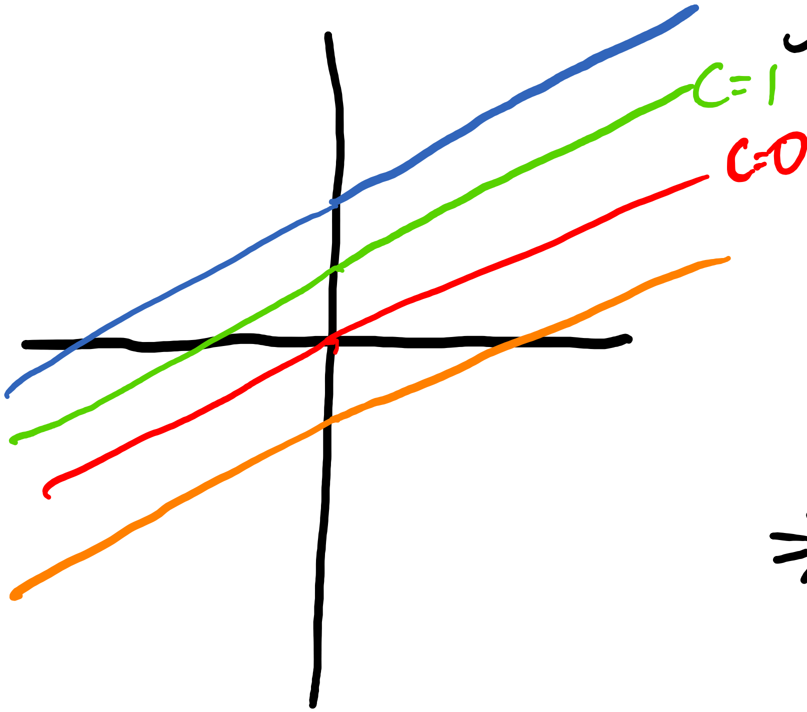
$$= \frac{1}{2}\sin 2x + \frac{5}{\pi}e^{\pi x} + C$$

Check: $\frac{d}{dx}\left[\frac{1}{2}\sin 2x + \frac{5}{\pi}e^{\pi x} + C\right] = \cos 2x + 5e^{\pi x}$ ✓

Initial value problems and antiderivatives

- Recall that when there are infinitely many antiderivatives for a function. We can choose a specific function by giving an initial/boundary value.

Find the function f such that (s.t.) $f'(x) = \frac{1}{2}$ and $f(0) = 1$.



$$\int f'(x)\,dx = \frac{x}{2} + C$$

$$f(x) = \frac{x}{2} + C$$

$$f(0) = \frac{0}{2} + C = 1$$

$$\Rightarrow C = 1$$

$$\Rightarrow f(x) = \frac{x}{2} + 1$$

Example - IVP

- Consider the function f such that $f'(x) = x^2 + 2x$ where $f(0) = 0$.

integration rules

$$f(x) = \int (x^2 + 2x)\,dx = \frac{1}{3}x^3 + x^2 + C$$

$$f(0) = 0 = \frac{1}{3}\cdot 0^3 + 0^2 + C$$

$$\Rightarrow \quad C = 0$$

$$f(x) = \frac{1}{3}x^3 + x^2$$

$$f(1) = \frac{1}{3}\cdot 1^3 + 1^2 = \frac{4}{3}$$

What is $f(1)$?

A: $\frac{4}{3}$
B: $\frac{2}{3}$
C: 1
D: $x^2 + C$
E: None of the above