Integration and antiderivatives Lecture 1a: 2023-01-09

MAT A02 – Winter 2023 – UTSC

Prof. Yun William Yu

Survey of backgrounds

- Type in chat “a”, “b”, “c”, “d”, or “e” to state your field of interest:
    - A: Biology
    - B: Chemistry
    - C: Mathematics
    - D: Environmental Science
    - E: Other
- You may also add “?” to pump up the confusion meter to test out that functionality (or if you’re confused).
- The “!” corresponds to your excitement level.

Re-invention of the trapezoid rule

A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves

MARY M. TAI, MS, EDD

OBJECTIVE — To develop a mathematical model for the determination of total areas under curves from various metabolic studies.

RESEARCH DESIGN AND METHODS — In Tai's Model, the total area under a curve is computed by dividing the area under the curve between two designated values on the *X*-axis (abscissas) into small segments (rectangles and triangles) whose areas can be accurately calculated from their respective geometrical formulas. The total sum of these individual areas thus represents the total area under the curve. Validity of the model is established by comparing total areas obtained from this model to these same areas obtained from graphic method (less than ±0.4%). Other formulas widely applied by researchers under- or overestimated total area under a metabolic curve by a great margin.

RESULTS — Tai's model proves to be able to *1*) determine total area under a curve with precision; *2*) calculate area with varied shapes that may or may not intercept on one or both *X/Y* axes; *3*) estimate total area under a curve plotted against varied time intervals (abscissas), whereas other formulas only allow the same time interval; and *4*) compare total areas of metabolic curves produced by different studies.

CONCLUSIONS — The Tai model allows flexibility in experimental conditions, which means, in the case of the glucose-response curve, samples can be taken with differing time intervals and total area under the curve can still be determined with precision.

Estimation of total areas under curves of metabolic studies has become an ... under a glucose-tolerance or an energy-expenditure curve (1,2) ...

ating ...
resea...

From ...
Depa...
Ad...
Nutri...
NY 1...
Rec...
1993.

However, except for Wolever et al.'s formula, other formulas tend to under- or overestimate the total area under a metabolic curve by a large margin.

RESEARCH DESIGN AND METHODS

Tai's mathematical model

Tai's model was developed to correct the deficiency of under- or overestimation of the total area under a metabolic curve. This formula also allows calculating the area under a curve with unequal units on the *X*-axis. The strategy of this mathematical model is to divide the total area under a curve into individual small segments such as squares, rectangles, and triangles, whose areas can be precisely determined according to existing geometric formulas. The area of the individual segments are then added to obtain the total area under the curve. As shown in Fig. 1, the total area can be expressed as:

Total area = triangle *a* + rectangle *b* + triangle *c* + rectangle *d* + triangle *e* + rectangle *f* + triangle *g* + rectangle *h* + ...

If y = height, x = width

Area (square) = x^2 or y^2 ($x = y$);

Area (rectangle) = xy;

Area (triangle) = $xy/2$

Let: $X_1 = x_2 - x_1$; $X_2 = x_3 - x_2$

$X_3 = x_4 - x_3$; $X_4 = x_5 - x_4$;

$X_{n-1} = x_n - x_{n-1}$

Total Area = $\frac{1}{2}X_1(y_2 - y_1) + X_1y_1 +$

$\frac{1}{2}X_2(y_3 - y_2) + X_2y_2 +$

$\frac{1}{2}X_3(y_4 - y_3) + X_3y_3$

... $- y_4) + X_4y_4 + \ldots$

... $(y_n - y_{n-1}) + X_{n-1}y_{n-1}$

... $+ X_1y_2 + X_2y_2 + X_2y_3 + X_3y_3 +$

... $y_4 + X_4y_5 + \ldots + X_{n-1}y_{n-1}$

... $= \frac{1}{2}[X_1(y_1 + y_2) + X_2(y_2 + y_3)$

... $+ y_4) + X_4(y_4 + y_5) + \ldots$

... $+ Y_n)]$

...ve passes the origin, $1/2[X_0y_1]$ ... added to above formula. If the ...rcepts at y_0 at the *Y*-axis, let ... x_0, $1/2[X_0(y_0 + y_1)]$ should be ...e above formula; Tai's formula ... different conditions.

How long ago was the trapezoid rule invented?

A: Before 1 AD

B: 1-1000 AD

C: 1000-1500 AD

D: 1500-1900 AD

E: 1900-2000 AD

Differentiation and dimensional analysis

- Gets instantaneous rate of change of one variable vs another.
  - Let the variable x represent distance along the x-axis (in meters) and $y = f(x)$ be the height of the function on the y-axis (in meters).
    - Then $\frac{df}{dx} = f'(x)$ is the instantaneous slope (meters / meters = unitless).
  - Let the variable t represent time (in sec.) and the variable $x = x(t)$ be distanced traveled along the x-axis (in meters).
    - Then $\frac{dx}{dt} = \dot{x}(t)$ is the velocity (in meters/second)
  - Let the $v(t) = \dot{x}(t)$ be the velocity (meters/second, m/s)
    - Then $\frac{dv}{dt} = \ddot{x}(t)$ is the acceleration (in m/s^2)
  - Let $P(t)$ be the population (unitless count) at time t.
    - Then $\frac{dP}{dt}$ is the population growth rate (in number / second)

Integration

- “Opposite” of differentiation
  - “Summing up value as another variable changes”
  - Dimensional analysis: multiply together units along axes

Antiderivatives

Theorem: If two functions F and G have the same derivative $F'(x) = G'(x)$, then $F(x) = G(x) + C$, where C is a constant.

Can reverse many differentiation rules

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\,dx = \ln\|x\| + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\;\,dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\;\,dx = \frac{1}{\mathrm{a}}\sin ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\tan ax\right] = \sec^2 ax$	$\int \sec^2 ax\;\,dx = \frac{1}{\mathrm{a}}\tan ax + C$
$\frac{d}{dx}\left[-\frac{1}{\mathrm{a}}\cot ax\right] = \csc^2 ax$	$\int \csc^2 ax\;\,dx = -\frac{1}{\mathrm{a}}\cot ax + C$
$\frac{d}{dx}\left[\frac{1}{a}\sec ax\right] = \sec ax \tan ax$	$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec x + C$
$\frac{d}{dx}\left[-\frac{1}{a}\csc ax\right] = \csc ax \cot ax$	$\int \csc ax \cot ax\,dx = -\frac{1}{\mathrm{a}}\csc ax + C$

Advanced rules

- Constant multiplication rule

$$\int kf(x)dx = k\int f(x)dx$$

- Addition/subtraction rule

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$
$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

- No product or quotient rule. See: Integration by Parts.

Examples

$\int (ma + a^{35})da$, where m is a constant.

A: $m + 35a^{34}$
B: $m + 35a^{34} + C$
C: $\frac{m}{2}a^2 + \frac{1}{36}a^{36}$
D: $\frac{m}{2}a^2 + \frac{1}{36}a^{36} + C$
E: None of the above

Examples

Initial value problems and antiderivatives

- Recall that when there are infinitely many antiderivatives for a function. We can choose a specific function by giving an initial/boundary value.

Example - IVP

- Consider the function f such that $f'(x) = x^2 + 2x$ where $f(0) = 0$.

What is $f(1)$?

A: $\frac{4}{3}$

B: $\frac{2}{3}$

C: 1

D: $x^2 + C$

E: None of the above