Area under curves and the Fundamental Theorem of Calculus Lecture 1b: 2023-01-09

MAT A02 – Winter 2023 – UTSC

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Can reverse many differentiation rules

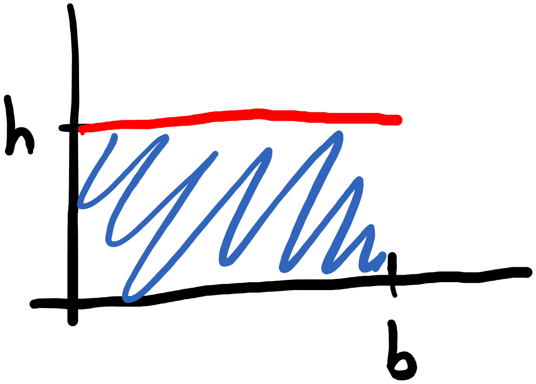
Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\,dx = \ln\|x\| + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\ dx = \frac{1}{\mathrm{a}}\sin ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\tan ax\right] = \sec^2 ax$	$\int \sec^2 ax\ dx = \frac{1}{\mathrm{a}}\tan ax + C$
$\frac{d}{dx}\left[-\frac{1}{\mathrm{a}}\cot ax\right] = \csc^2 ax$	$\int \csc^2 ax\ dx = -\frac{1}{\mathrm{a}}\cot ax + C$
$\frac{d}{dx}\left[\frac{1}{a}\sec ax\right] = \sec ax \tan ax$	$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec x + C$
$\frac{d}{dx}\left[-\frac{1}{a}\csc ax\right] = \csc ax \cot ax$	$\int \csc ax \cot ax\,dx = -\frac{1}{\mathrm{a}}\csc ax + C$

memorize

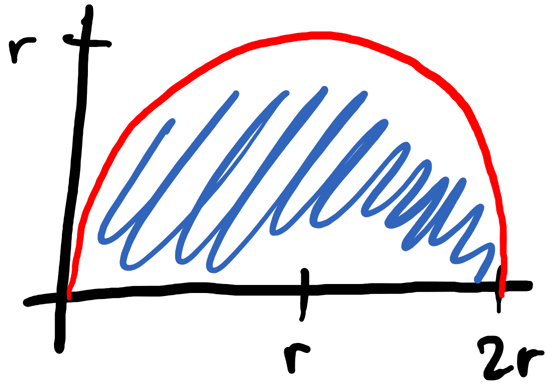
will be provided

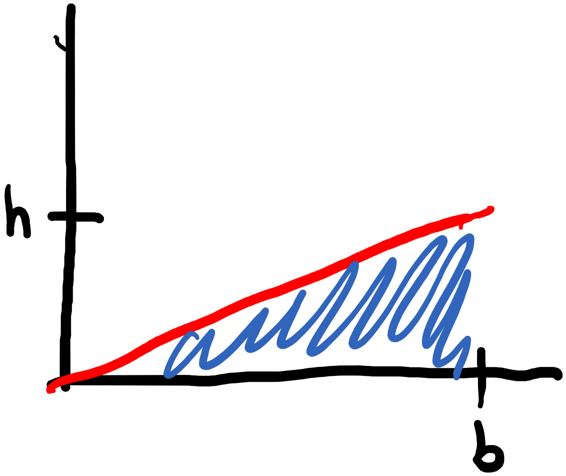
Area under curve

$AUC = \frac{1}{2}\pi r^2$

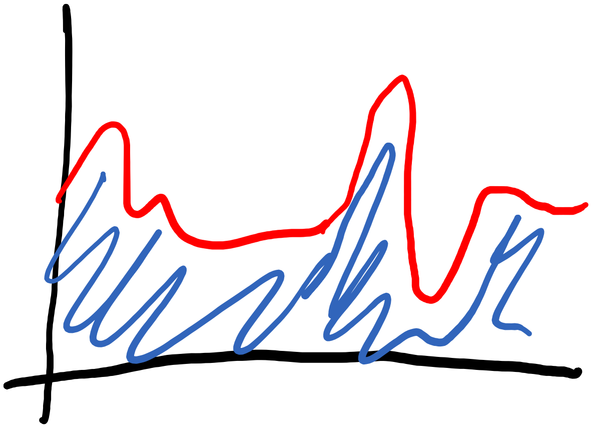


$AUC = bh$





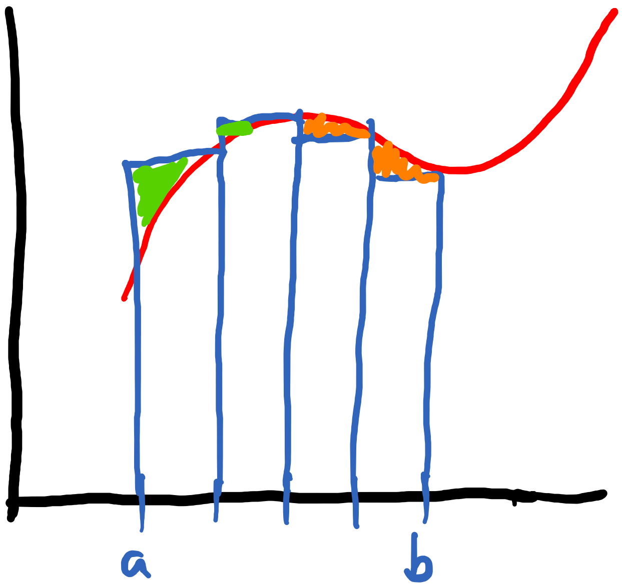
$AUC = \frac{1}{2}bh$

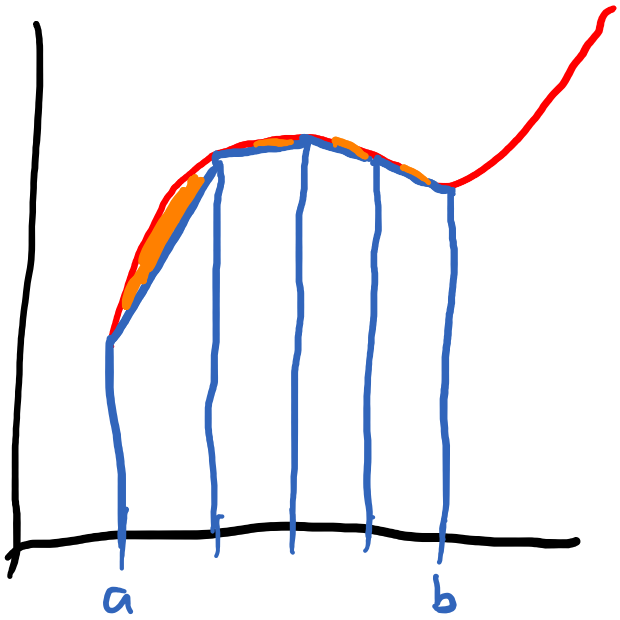


AUC = ?

Riemann sums and trapezoid rule

- We can approximate area under any curve by dividing into shapes we know how to compute area for, like rectangles or trapezoids

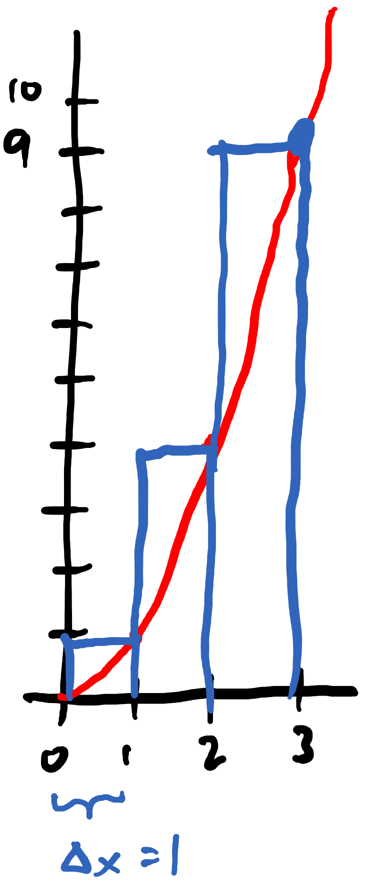




One way of reducing error is by choosing better shapes

Example

- Approximate the area under the parabola $y = x^2$ between 0 and 3 using a Riemann sum with 3 rectangles.



$\Delta x = 1$

$f(0) = 0$

$f(1) = 1$

$f(2) = 4$

$f(3) = 9$

Area:

$\approx 1\Delta x + 4\Delta x + 9\Delta x$

$= 14$

Try it out

- Approximate the area under the line $y = x$ between 0 and 4 using a Riemann sum.

Use 2 rectangles

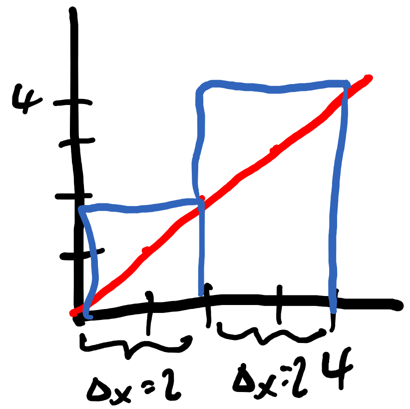
A: 8
B: 10
C: 12
D: 14
E: None

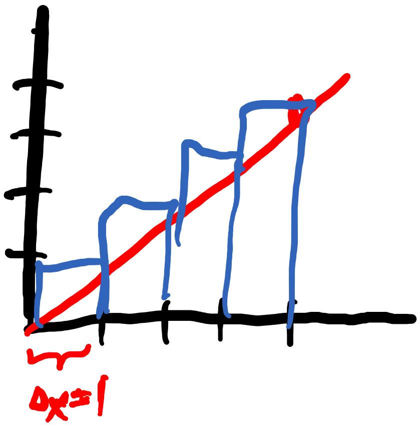
Use 4 rectangles

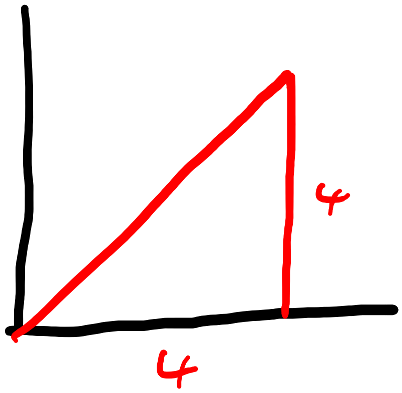
A: 8
B: 10
C: 12
D: 14
E: None

Actual area

A: 8
B: 10
C: 12
D: 14
E: None







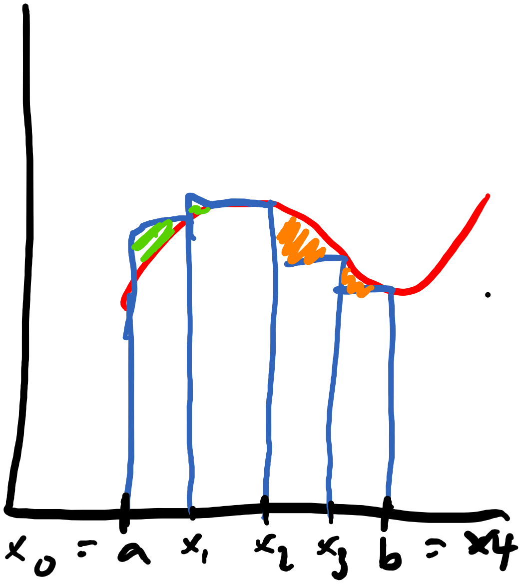
$2 \cdot 2 + 2 \cdot 4 = 4 + 8 = 12$

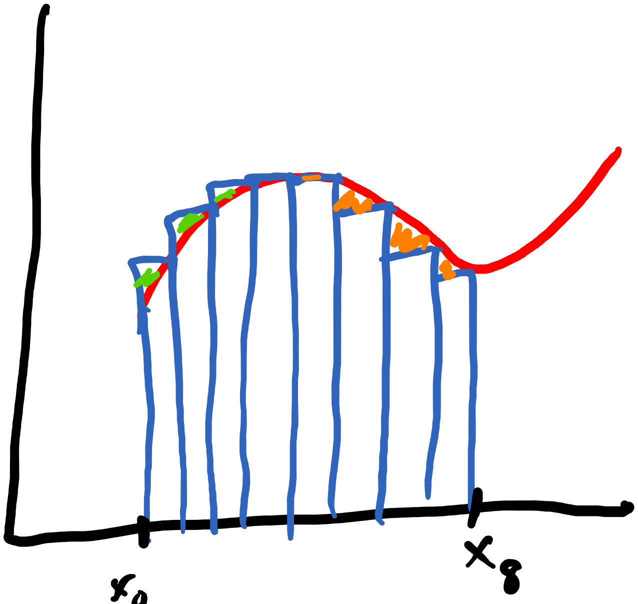
$1 + 2 + 3 + 4 = 10$

$\frac{1}{2} \cdot 4 \cdot 4 = 8$

More rectangles

- Another way to decrease approximation error is to use more rectangles.





Let $\Delta x = \frac{1}{4}(b-a)$

$\Delta x = \frac{1}{8}(b-a)$

Let $x_i = a + i\Delta x$ for $i = 0, 1, 2, 3, 4$

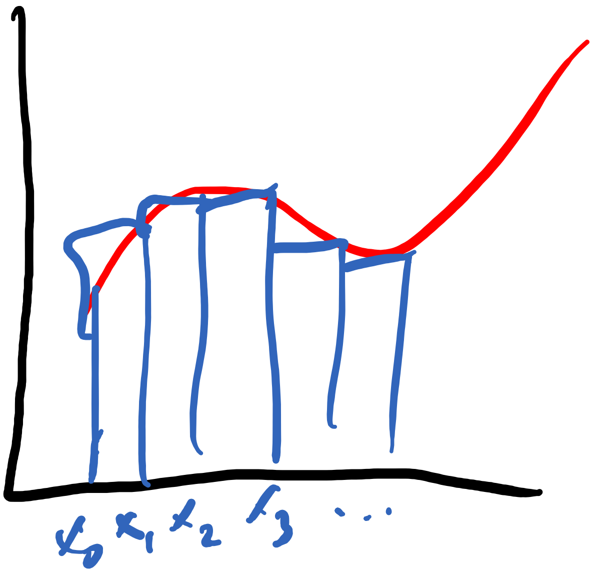
$x_i = a + i\Delta x$ for i from 0 to 8

Area $\approx \sum_{i=1}^{4} \Delta x \cdot f(x_i)$

Area $\approx \sum_{i=1}^{8} \Delta x \cdot f(x_i)$

Infinite rectangles!

- Take the limit as the rectangles become infinitely thin.



Let $\Delta x = \frac{1}{n}(b-a)$

And $x_i = a + i \Delta x$

Define $\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i) \Delta x$

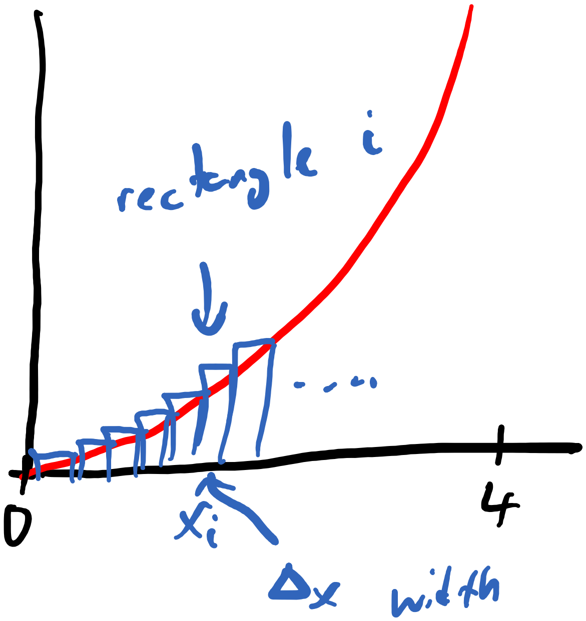
Definition: Let f be a continuous function on $[a, b]$ with $a < b$. Then the definite integral of f from a to b is defined by

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i) \Delta x$$

where $\Delta x = \frac{1}{n}(b-a)$ and $x_i = a + i\Delta x$. a and b are the *limits of integration*. If $f(x) > 0$ on $[a, b]$, then the definite integral represents the area between the curve $y = f(x)$ and the x-axis.

Riemann sum example: $\int_0^4 x^2 \, dx$

$\Delta x = \frac{4-0}{n} = \frac{4}{n}$ Let $f(x) = x^2$



$f(x_i) = x_i^2$

$x_0 = 0$

$x_1 = \Delta x = \frac{4}{n}$

$x_2 = 2\Delta x = \frac{8}{n}$

$\vdots$

$x_i = i\Delta x = \frac{4i}{n}$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x$$

$$= \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^2 \cdot \frac{4}{n}$$

$$= \left(\frac{4}{n}\right)^3 \cdot \sum_{i=1}^{n} i^2$$

$$= \left(\frac{4}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6} = \frac{32(n+1)(2n+1)}{3n^2}$$

$$\Rightarrow \int_0^4 x^2 \, dx = \lim_{n\to\infty} \frac{32(n+1)(2n+1)}{3n^2} = \frac{32}{3} \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2}$$

$$= \frac{32}{3} \cdot 2 = \frac{64}{3}.$$

Signed Area

The definite integral gives a signed area, which is positive when the function is positive and negative when the function is negative.

Positive or negative?

A: Positive

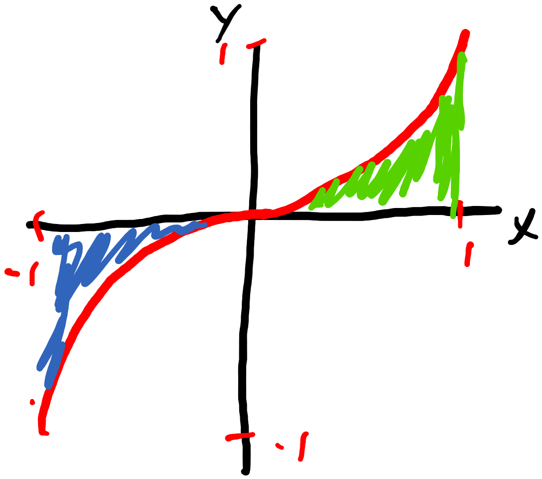
B: 0

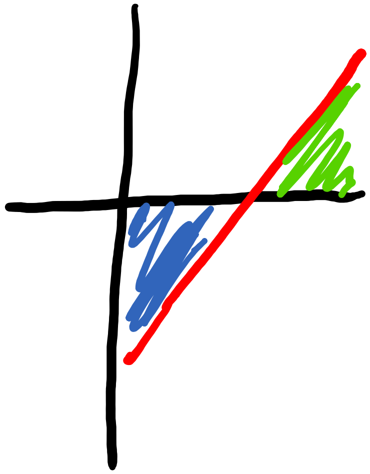
C: Negative

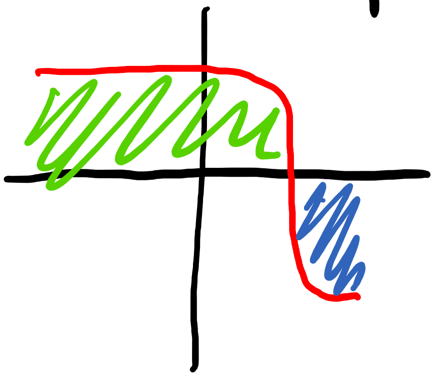
D: Neither

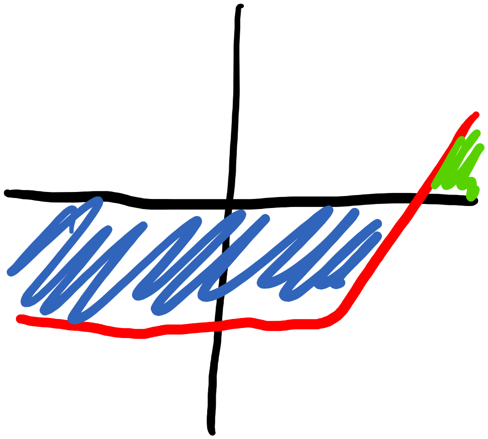
E: Didn’t pay attention

$$y = \int_{-1}^{1} x^3 \, dx$$









Fundamental Theorem of Calculus

- First form of the Fundamental Theorem of Calculus
  - Let f be a continuous function and let $A(x) = \int_{a}^{x} f(t)dt$. Then $A'(x) = f(x)$

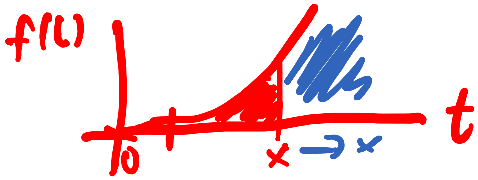
    variable

    constant
  - If you integrate a function and then take the derivative, you get the same function back.
- Second form of the Fundamental Theorem of Calculus
  - Let $f(x)$ be a continuous function and suppose that $g'(x) = f(x)$ (i.e. $g(x)$ is an antiderivative of $f(x)$). Then $\int_{a}^{b} f(x)dx = g(b) - g(a)$
  - You can use the antiderivative of a function to compute the definite integral without explicitly using infinite Riemann sums.

Note: $\int f(x)dx = g(x) + C$ ← unknown

$\int_a^b f(x)\,dx = g(b) - g(a)$ ← definite, no unknown

Example



$$\frac{d}{dx}\int_{t=0}^{t=x} t\sin^2 t\,dt = x\sin^2 x \qquad \text{(FTC form 1)}$$

$$\frac{d}{dy}\int_{x=-1000}^{x=y} e^{-x^2}\,dx = e^{-y^2} \qquad \text{(FTC form 1)}$$

$$\int_{x=0}^{x=2} e^{x}\,dx = e^{x}\Big|_{x=0}^{x=2} = e^{2}-e^{0} = e^{2}-1 \qquad \text{(FTC form 2)}$$

$$\int_{x=\frac{\pi}{2}}^{x=\pi} \sin x\,dx = -\cos x\Big|_{\frac{\pi}{2}}^{\pi} = \left(-\cos\pi\right) - \left(-\cos\frac{\pi}{2}\right) = -(-1)-0$$

$$= 1$$

FTC form 2

$$\frac{d}{dx}\int_{0}^{x} \sin t\,dt = \frac{d}{dx}\Big[-\cos t\Big]_{t=0}^{t=x}$$

$$= \frac{d}{dx}\Big[-\cos x + 1\Big] = \sin x$$

Application

cells / hour

$\int$ over hours

cells

- Bacteria in a petri dish grow at a rate of $P'(t) = 100e^{-t}$ cells per hour, where t is time in hours. Determine how much the population increases from time $t = 0$ to time $t = 2$.

$x^0 = 1$ for any x

$e^0 = 1$

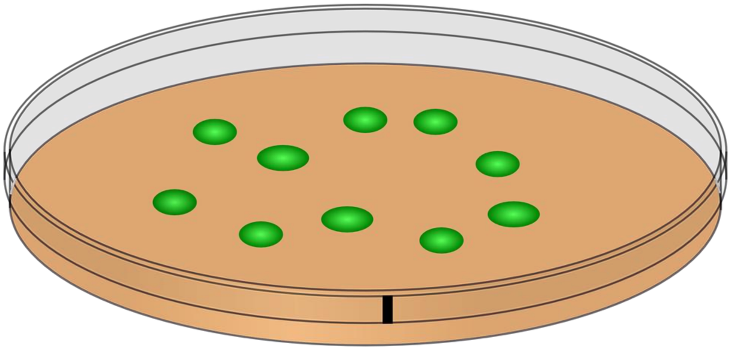
$$\int_{t=0}^{t=2} 100e^{-t}\,dt = 100\int_0^2 e^{-t}\,dt$$

$$= 100\left[-e^{-t}\right]_{t=0}^{t=2}$$

$$= 100\left[-e^{-2} - (-1)\right]$$

$$= 100\left[1 - e^{-2}\right] \longleftarrow$$

$$\approx 86.466$$



Application

- Corn needs 1.5 inches of rainfall or watering per week.
- Suppose it rains today between noon and 1pm at a rate of $f(t) = 2 - t^2$ inches/hour, where t is the number of hours since noon.
- Did it rain enough that you do not need to water your corn field?

$$\int_0^1 f(t)\,dt = \int_0^1 (2-t^2)\,dt = \left[2t - \frac{1}{3}t^3\right]_{t=0}^{t=1}$$

$$= 2 - \frac{1}{3} = 1.667 \text{ inches}$$

A: Yes
B: Maybe
C: No
D: No clue
E: ???