Area under curves and the Fundamental Theorem of Calculus Lecture 1b: 2023-01-09

MAT A02 – Winter 2023 – UTSC

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Can reverse many differentiation rules

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\, dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\, dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\|x\|] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\, dx = \ln\|x\| + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\;\, dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\;\, dx = \frac{1}{\mathrm{a}}\sin ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\tan ax\right] = \sec^2 ax$	$\int \sec^2 ax\;\, dx = \frac{1}{\mathrm{a}}\tan ax + C$
$\frac{d}{dx}\left[-\frac{1}{\mathrm{a}}\cot ax\right] = \csc^2 ax$	$\int \csc^2 ax\;\, dx = -\frac{1}{\mathrm{a}}\cot ax + C$
$\frac{d}{dx}\left[\frac{1}{a}\sec ax\right] = \sec ax \tan ax$	$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec x + C$
$\frac{d}{dx}\left[-\frac{1}{a}\csc ax\right] = \csc ax \cot ax$	$\int \csc ax \cot ax\, dx = -\frac{1}{\mathrm{a}}\csc ax + C$

Area under curve

Riemann sums and trapezoid rule

- We can approximate area under any curve by dividing into shapes we know how to compute area for, like rectangles or trapezoids

Example

- Approximate the area under the parabola $y = x^2$ between 0 and 3 using a Riemann sum with 3 rectangles.

Try it out

- Approximate the area under the line $y = x$ between 0 and 4 using a Riemann sum.

Use 2 rectangles

A: 8
B: 10
C: 12
D: 14
E: None

Use 4 rectangles

A: 8
B: 10
C: 12
D: 14
E: None

Actual area

A: 8
B: 10
C: 12
D: 14
E: None

More rectangles

- Another way to decrease approximation error is to use more rectangles.

Infinite rectangles!

- Take the limit as the rectangles become infinitely thin.

Definition: Let f be a continuous function on $[a, b]$ with $a < b$. Then the definite integral of f from a to b is defined by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)$$

where $\Delta x = \frac{1}{n}(b-a)$ and $x_i = a + i\Delta x$. a and b are the *limits of integration*. If $f(x) > 0$ on $[a, b]$, then the definite integral represents the area between the curve $y = f(x)$ and the x-axis.

Riemann sum example: $\int_0^4 x^2 dx$

Signed Area

The definite integral gives a signed area, which is positive when the function is positive and negative when the function is negative.

Positive or negative?

A: Positive

B: 0

C: Negative

D: Neither

E: Didn’t pay attention

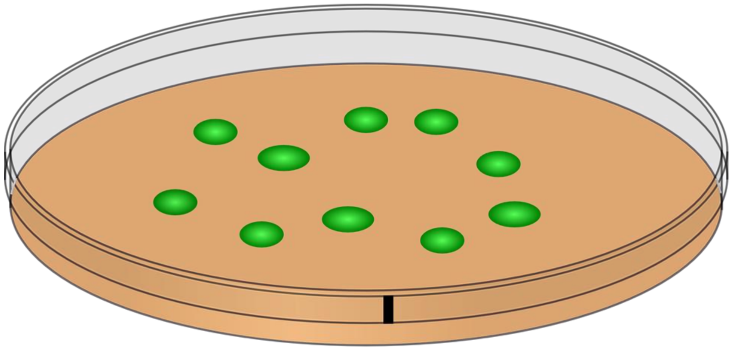
Fundamental Theorem of Calculus

- First form of the Fundamental Theorem of Calculus
  - Let f be a continuous function and let $A(x) = \int_a^x f(t)dt$. Then $A'(x) = f(x)$
  - If you integrate a function and then take the derivative, you get the same function back.
- Second form of the Fundamental Theorem of Calculus
  - Let $f(x)$ be a continuous function and suppose that $g'(x) = f(x)$ (i.e. $g(x)$ is an antiderivative of $f(x)$). Then $\int_a^b f(x)dx = g(b) - g(a)$
  - You can use the antiderivative of a function to compute the definite integral without explicitly using infinite Riemann sums.

Example

Application

- Bacteria in a petri dish grow at a rate of $P'(t) = 100e^{-t}$ cells per hour, where t is time in hours. Determine how much the population increases from time $t = 0$ to time $t = 2$.



Application

- Corn needs 1.5 inches of rainfall or watering per week.
- Suppose it rains today between noon and 1pm at a rate of $f(t) = 2 - t^2$ inches/hour, where t is the number of hours since noon.
- Did it rain enough that you do not need to water your corn field?

A: Yes
B: Maybe
C: No
D: No clue
E: ???