# More advanced integration
# Lecture 1c: 2023-01-12

MAT A02 – Winter 2023 – UTSC

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# Average of a function

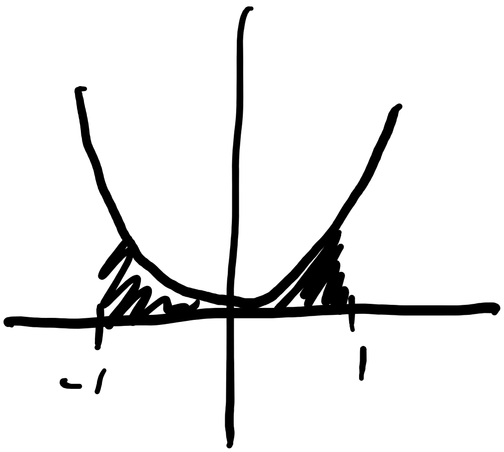
- Let $f:[\mathrm{a},\mathrm{b}] \rightarrow \mathbb{R}$ be a continuous function. Then its average value $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$.

Ex. $f(x) = x^2$

Find average value b/t $-1$ and $1$.

$$y_{av} = \frac{1}{\underbrace{(1-(-1))}_{2}} \int_{-1}^{1} x^2\, dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]\Bigg|_{x=-1}^{x=1}$$

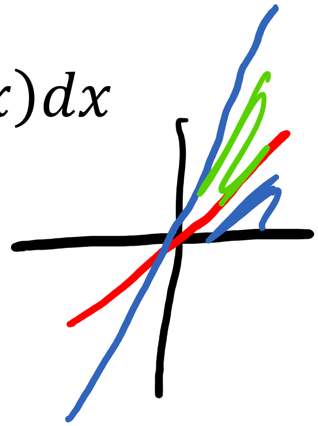
$$= \frac{1}{6}\left[1^3 - (-1)^3\right] = \underbrace{\frac{1}{3}}_{\text{avg value}}$$



# Properties of definite integrals

- Constant multiplication: $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$

$$\int_0^1 2 \cdot x \, dx = 2\int_0^1 x \, dx$$

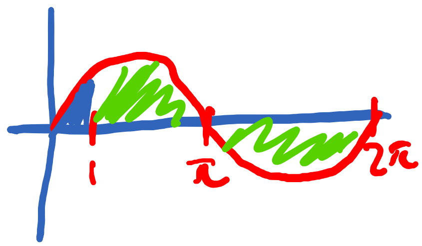


- Sum of different integrands with same bounds
  - $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$$\int_0^1 (1+x)\,dx = \int_0^1 1\,dx + \int_0^1 x\,dx$$

- Sum of same integrand with touching bounds
  - $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ where $a < b < c$

$$\int_0^1 \sin x\,dx + \int_1^{2\pi} \sin x\,dx = \int_0^{2\pi} \sin x\,dx$$



# Try it out

$$= \int_1^2 + \int_2^3 - \int_1^2$$

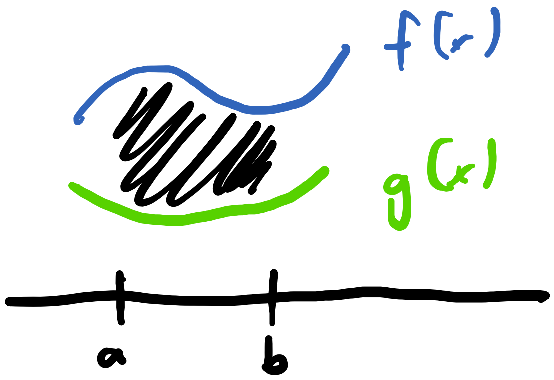
$$\underbrace{\int_0^2 x^2 dx + \int_0^2 5dx}_{\int_0^2 (x^2+5)dx} + \underbrace{\int_1^3 (x^2+5)dx - \int_1^2 (x^2+5)dx}_{\int_2^3 (x^2+5)dx}$$

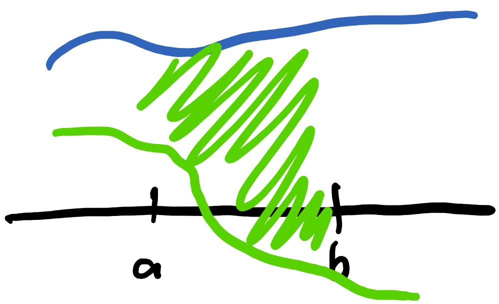
$$= \int_0^3 (x^2+5)dx = \left[\frac{1}{3}x^3 + 5x\right]_0^3 = 9+15 = 24$$

A: 24
B: 27
C: 30
D: 33
E: None of the above

# Area between curves

Let $f$ and $g$ be continuous functions, and suppose that $f(x) \geq g(x)$ over the interval $[a, b]$. Then the area of the region between the two curves on that interval is $\int_a^b [f(x) - g(x)]dx$.



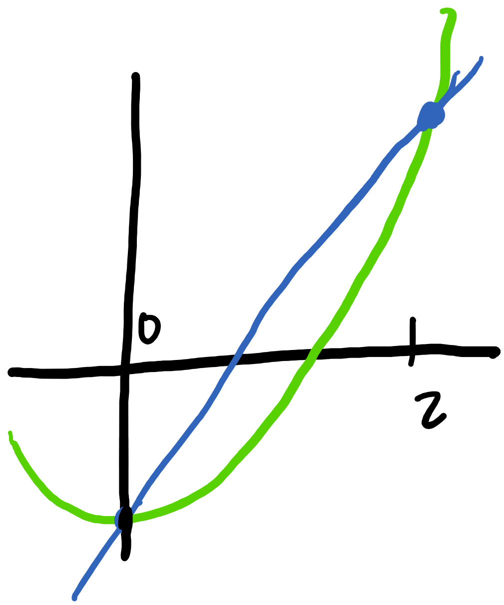

When $[a, b]$ are unknown, can compute the intersection points to figure out the area bounded by curves.

# Example

- Find the area bounded by the graphs of
  $$f(x) = 2x - 2 \text{ and } g(x) = x^2 - 2.$$

Intersections $f(x) = g(x)$

$$\Rightarrow x^2 - 2 = 2x - 2$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0, 2$$




Area: $\int_0^2 [f(x) - g(x)]dx$

$$= \int_0^2 (2x - x^2)dx$$
$$= \left[x^2 - \frac{1}{3}x^3\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

# Try it out

- Find the area bounded by graphs of $f(x) = x^2$ and $g(x) = x$.
- Step 1: find the intersection points.

$x^2 = x$

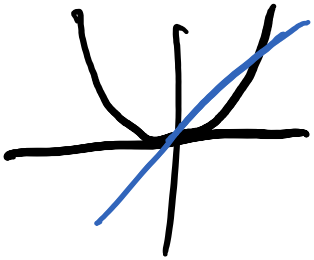
$\Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$\Rightarrow x = 0, 1$

A: -1, 1
B: 0, 2
C: -1, 0
D: 0, 1
E: None of the above

- Step 2: Decide which graph is on top.



$x > x^2$ for $x$ b/t 0 and 1

Also can check $f(0.5) = 0.25$, $g(0.5) = 0.5$

A: $f(x)$
B: $g(x)$
C: neither
D: both
E: ?????

- Step 3: Compute the integral.

$$\int_0^1 [x - x^2]\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

A: 1/3
B: 1/4
C: 1/5
D: 1/6
E: None of the above