More advanced integration
Lecture 1c: 2023-01-12

MAT A02 – Winter 2023 – UTSC

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Average of a function

- Let $f: [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ be a continuous function. Then its average value $y_{av} = \frac{1}{b-a} \int_a^b f(x) dx$.

Properties of definite integrals

- Constant multiplication: $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$

- Sum of different integrands with same bounds
  - $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

- Sum of same integrand with touching bounds
  - $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ where $a < b < c$

Try it out

$$\int_0^2 x^2 dx + \int_0^2 5 dx + \int_1^3 (x^2+5) dx - \int_1^2 (x^2+5) dx$$

A: 24
B: 27
C: 30
D: 33
E: None of the above

Area between curves

Let f and g be continuous functions, and suppose that $f(x) \geq g(x)$ over the interval $[a, b]$. Then the area of the region between the two curves on that interval is $\int_a^b [f(x) - g(x)]dx$.

When $[a, b]$ are unknown, can compute the intersection points to figure out the area bounded by curves.

Example

- Find the area bounded by the graphs of
  $f(x) = 2x - 2$ and $g(x) = x^2 - 2$.

Try it out

- Find the area bounded by graphs of $f(x) = x^2$ and $g(x) = x$.
- Step 1: find the intersection points.

A: -1, 1
B: 0, 2
C: -1, 0
D: 0, 1
E: None of the above

- Step 2: Decide which graph is on top.

A: $f(x)$
B: $g(x)$
C: neither
D: both
E: ?????

- Step 3: Compute the integral.

A: $1/3$
B: $1/4$
C: $1/5$
D: $1/6$
E: None of the above

Chain rule → Substitution rule

- Chain rule: Let $f = f(u)$ be a function of u and $u = u(x)$ be a function of x. Then $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

- “u-substitution” is the opposite of the chain rule.

Substitution rule algorithm

- Step 1: Guess an appropriate u
- Step 2: Compute du, dx, and x
- Step 3: Substitute in to get rid of all the x's
- Step 4: Integrate as a function of u
- Step 5: Convert back to x's

Example

Substitution for definite integrals

Try it out

- $\int_0^2 \frac{x}{(1+x^2)^2}\,dx$

A: 0
B: 0.2
C: 0.4
D: 0.6
E: None of the above

- $\int \tan x\,dx$. Hint: $\tan x = \frac{\sin x}{\cos x}$. Let $u = \cos x$

A: $\ln|\sin x|^2 + C$
B: $-\ln|\sin x| + C$
C: $\ln|\cos x|^2 + C$
D: $-\ln|\cos x| + C$
E: None of the above

Integration techniques – partial fractions

- Sometimes, it is easier to integrate if you break up a complicated expression into several simpler ones. One way to do this is with a partial fractions decomposition:
$$\frac{h(x)}{f(x)g(x)} = \frac{A(x)}{f(x)} + \frac{B(x)}{g(x)}$$
Where $h(x), f(x), g(x), A(x), B(x)$ are all polynomials in x.

Example

Try it out: $\int \frac{5x+1}{2x^2-x-1}\,dx$

1: Factor: $2x^2 - x - 1$

A: $(2x-1)(x-1)$
B: $(2x+1)(x-1)$
C: $(2x-1)(x+1)$
D: $(2x+1)(x+1)$
E: None of the above

2: Solve for $\frac{5x+1}{2x^2-x-1} = \frac{A}{2x+1} + \frac{B}{x-1}$

A: $A = 1, B = 1$
B: $A = 1, B = 2$
C: $A = 2, B = 2$
D: $A = 2, B = 1$
E: None of the above

3: Integrate $\int \frac{5x+1}{2x^2-x-1}\,dx = \int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$

Product Rule → Integration by parts

- Recall $\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x)$

- Integration by parts is the opposite of the product rule:
  - $\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$
  - $d[u(x)v(x)] = u \cdot dv + v \cdot du$
  - $u \cdot dv = d[u(x)v(x)] - v \cdot du$
  - $\int u \cdot dv = \int d[u(x)v(x)] - \int v \cdot du$
  - $\int u\, dv = uv - \int v\, du$

Integration by parts algorithm

- $\int u \, dv = uv - \int v \, du$
- Step 1: Guess which part is u and which part is dv
- Step 2: Apply the formula above and hope you can solve $\int v \, du$
- Step 3: If it doesn't, try again with a different guess for u and dv.

- Step ?: Give up if no guess seems to work. The integral might not be amenable to integration by parts.

Example ($\int u\, dv = uv - \int v\, du$)

Example ($\int u\,dv = uv - \int v\,du$)

Try it out: $\int x^2 e^x dx$

$$\int u\, dv = uv - \int v\, du$$

Write your answer in chat.