Volume and improper integration Lecture 2b: 2023-01-16

MAT A02 – Winter 2023 – UTSC

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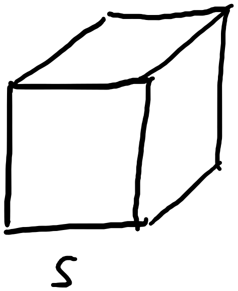
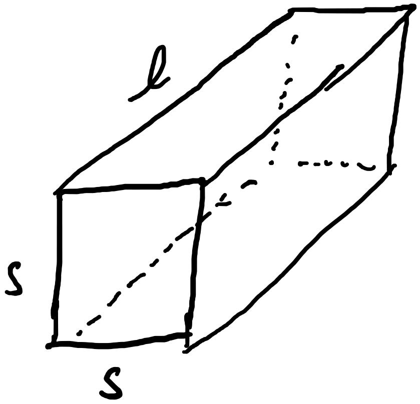
Volume of simple solids



cube

$Vol = s^3$

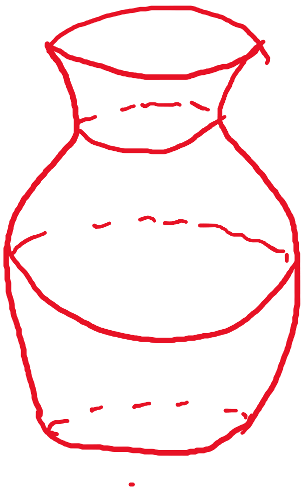


rectangular block

$Vol = \underbrace{s^2}_{\text{Area}}\ \underbrace{l}_{\text{Length}}$

cylinder

$Vol = \underbrace{\pi r^2}_{\text{Area}} \times \underbrace{h}_{\text{height}}$

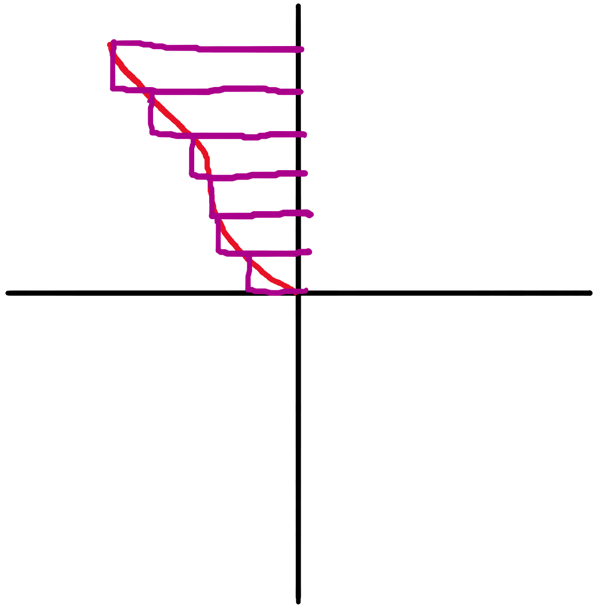


Vase ?

Invention of pottery?

A: Before 1 AD
B: 1-1000 AD
C: 1000-1500 AD
D: 1500-1800 AD
E: 1800 AD-present

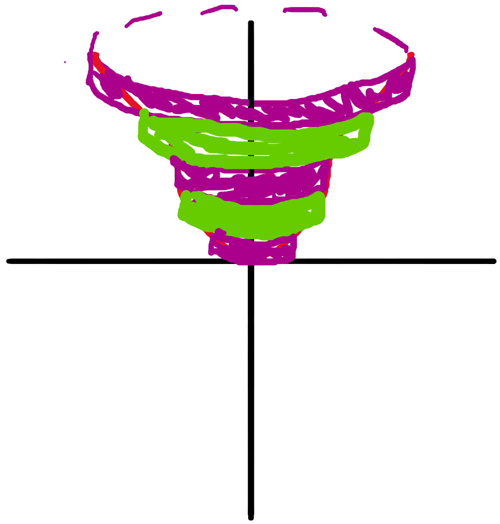
Solids of revolution



- Area under a curve can be approximated by rectangles

$$A = \lim_{n\to\infty} \sum_{1}^{n} f(x_i)\Delta x$$

$$= \int_a^b f(x)\,dx$$

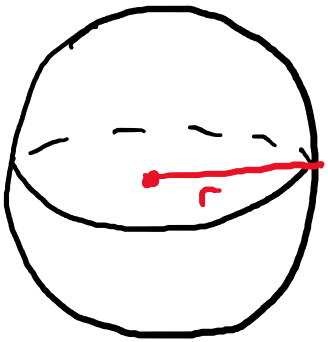


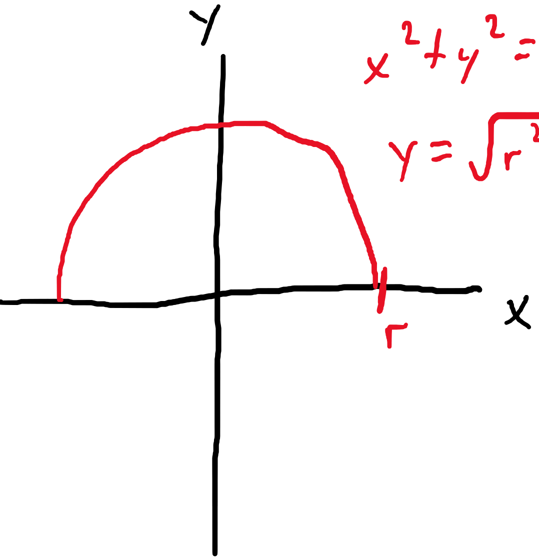
- What if we rotate about the vertical axis? What is the volume?

$$V = \lim_{n\to\infty} \sum_{1}^{n} \pi\big(f(x_i)\big)^2 \Delta x$$

$$= \int_a^b \pi\big(f(x)\big)^2\,dx$$

Example – Volume of a sphere





$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2} = f(x)$

$$\int_{-r}^{r} \pi \left(f(x)\right)^2 \, dx$$

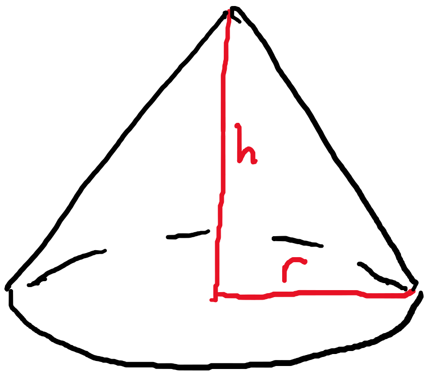
$$= 2\pi \int_0^r (r^2 - x^2)\, dx$$

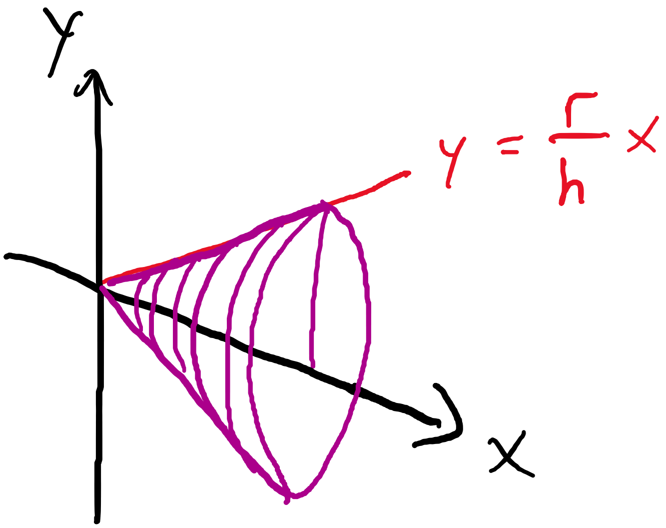
$$= 2\pi \left[r^2 x - \frac{1}{3} x^3 \right] \Big|_0^r$$

$$= 2\pi \left[r^3 - \frac{1}{3} r^3 \right] = 2\pi \cdot \frac{2}{3} r^3$$

$$= \boxed{\frac{4}{3} \pi r^3}$$

Example – Volume of cone





$$\int_0^h \pi\left(\frac{r}{h}\cdot x\right)^2 dx$$

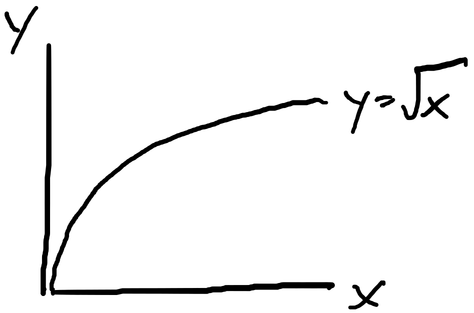
$$=\frac{\pi r^2}{h^2}\cdot\int_0^h x^2\,dx$$

$$=\frac{\pi r^2}{h^2}\cdot\left[\frac{1}{3}x^3\right]\Bigg|_0^h$$

$$=\boxed{\frac{1}{3}\pi r^2 h}$$

Try it out

- Find the volume of the solid of revolution generated by rotating the region under the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



$$\int_0^1 \pi (\sqrt{x})^2 \, dx$$

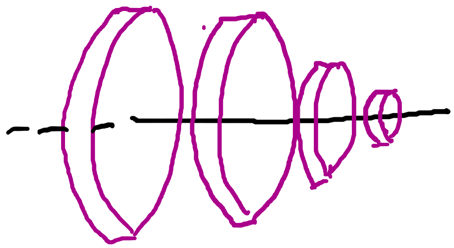
$$= \int_0^1 \pi x \, dx = \pi \cdot \frac{1}{2} x^2 \Big|_0^1 = \frac{\pi}{2}.$$

A: $\pi - 1$
B: $\pi/2$
C: $\pi/3$
D: π
E: None

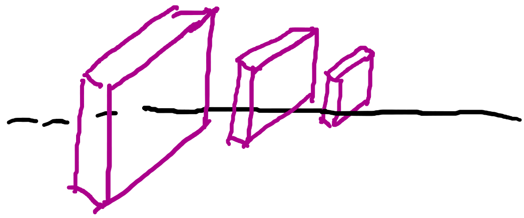
Other Volume Integrals

- Integrating disc volumes along an axis

$\lim_{n\to\infty} \sum_1^n \pi\left(f(x_i)\right)^2 \Delta x$



$= \int \pi(f(x))^2 \, dx$



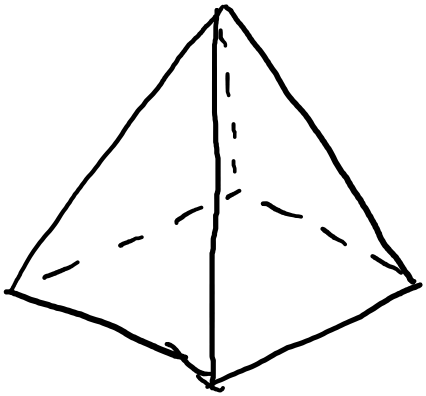
- What about other shapes?

$\lim_{n\to\infty} \sum_1^n A(x)\Delta x$, where $A(x)$ is the area of each slice to be multiplied by Δx.

$= \int A(x) \, dx$

Example - Pyramid

$$\int_a^b f(x)\,dx = F(b) - F(a) = -(F(a) - F(b)) = \int_b^a f(x)\,dx.$$



- Suppose the vertical cross section of a pyramid 100 meters tall is always a square, and suppose the side-length of the square is $100 - x$ meters, where x is the height above ground in meters.
- What is the volume of the pyramid?

$$A(x) = (100 - x)^2$$

$$\mathrm{Vol} = \int_{x=0}^{x=100} (100 - x)^2\,dx = -\int_{u=100}^{u=0} u^2\,du = \int_{u=0}^{u=100} u^2\,du$$

Let $u = 100 - x$

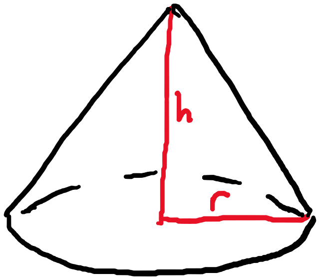
$du = -dx$

$$= \frac{1}{3} u^3 \Big|_{u=0}^{u=100} = \frac{1}{3} \cdot 100^3 = \frac{1000000}{3}$$

$$\approx 333333 \ m^3$$

Surface Areas?

πr^2 vs. $2\pi r$



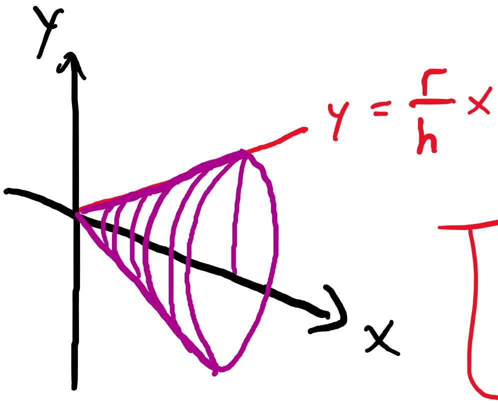
$$\text{Vol} = \int_0^h \pi \left(f(x)\right)^2 dx$$

$$\text{Surface Area} \overset{?}{=} \int_0^h 2\pi \left(f(x)\right) dx$$

(not including base)

A: True
B: False
C: ???
D: !!!
E: None

$\boxed{\text{FALSE}}$



$\boxed{\text{Correct Surface Area}}$

$$= \int_0^h 2\pi \left(f(x)\right)\sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2}\, dx$$

- Using discs does NOT work for surface areas because you are incorrectly approximating the paths.