

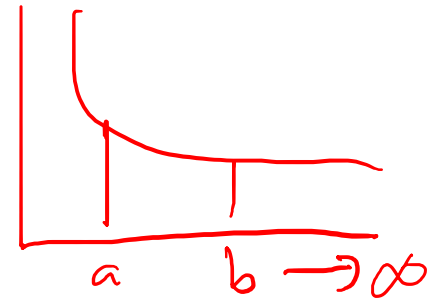
Improper integrals

Lecture 2c: 2023-01-19

MAT A02 – Winter 2023 – UTSC

Prof. Yun William Yu

Integrating to infinity



$$\bullet \int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

- If the limit exists, then the improper integral converges.
- If the limit does not exist, then the integral diverges.

$$\bullet \int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$



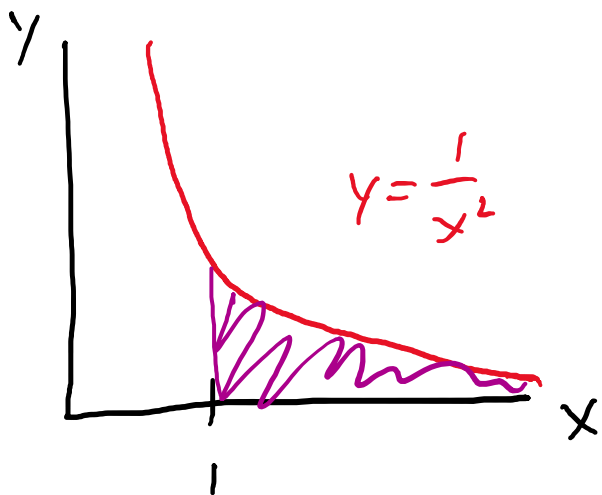
- If the limit exists, then the improper integral converges.
- If the limit does not exist, then the integral diverges.

$$\bullet \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx,$$

where c is any real number.

- If both integrals on the right converge, then so does the integral on the left.

Convergent Example



$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx$$

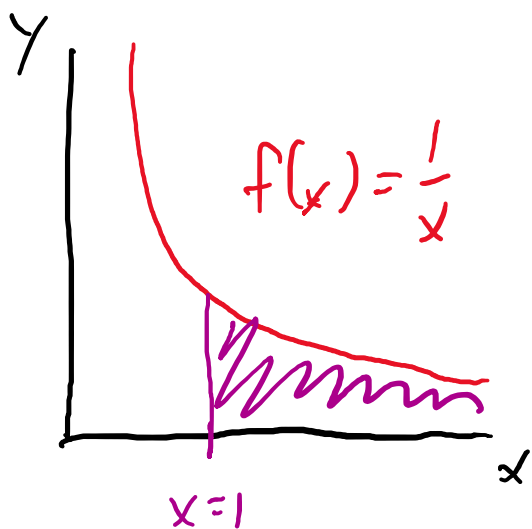
$$= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_{x=1}^{x=b}$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{1}{b} + 1 \right]$$

$$= 1$$

$$\left[-\frac{1}{b} \right] - \left[-\frac{1}{1} \right]$$

Divergent Example



$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx$$

$$= \lim_{b \rightarrow \infty} \left[\ln |x| \right]_{x=1}^{x=b}$$

$$= \lim_{b \rightarrow \infty} \left[\ln b - \ln 1 \right]$$

$$= \lim_{b \rightarrow \infty} \ln b = \infty$$

Example

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2} dx = \underbrace{\int_{-\infty}^0 \frac{x}{(x^2+1)^2} dx}_{\text{red underline}} + \underbrace{\int_0^{\infty} \frac{x}{(x^2+1)^2} dx}_{\text{green underline}}$$

$$\int_{-\infty}^0 \frac{x}{(x^2+1)^2} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{x}{(x^2+1)^2} dx = \lim_{a \rightarrow -\infty} \int_{a^2+1}^1 \frac{\frac{1}{2}}{u^2} du$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$= \lim_{a \rightarrow -\infty} \frac{1}{2} \left[-\frac{1}{u} \right] \Big|_{u=a^2+1}^{u=1} = \lim_{a \rightarrow -\infty} \frac{1}{2} \left[-1 + \frac{1}{a^2+1} \right] = -\frac{1}{2}$$

Alternately $\int \frac{x}{(x^2+1)^2} dx = \int \frac{\frac{1}{2}}{u^2} du = \frac{1}{2} \cdot \left[-\frac{1}{u} \right] + C = -\frac{1}{2(x^2+1)} + C$

$$\Rightarrow \lim_{a \rightarrow -\infty} \int_a^0 \frac{x}{(x^2+1)^2} dx = \lim_{a \rightarrow -\infty} \left[-\frac{1}{2(x^2+1)} \right] \Big|_{x=a}^{x=0} = \lim_{a \rightarrow -\infty} \left[-\frac{1}{2} + \frac{1}{2(a^2+1)} \right] = -\frac{1}{2}$$

Example (continued)

$$\begin{aligned} \int_0^{\infty} \frac{x}{(x^2+1)^2} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2+1)^2} dx = \lim_{b \rightarrow \infty} \int_1^{b^2+1} \frac{\frac{1}{2}}{u^2} du \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \cdot -\frac{1}{u} \right] \Big|_{u=1}^{u=b^2+1} \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} \cdot \frac{1}{(b^2+1)} + \frac{1}{2} \right] = \frac{1}{2} \end{aligned}$$

$u = x^2 + 1$
 $du = 2x dx$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2} dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Try it out

$$\bullet \int_0^{\infty} e^{-x} dx$$

$$\begin{aligned} &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-b} + 1 \right] = 0 \end{aligned}$$

A: Convergent
B: Divergent
E: Neither

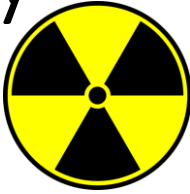
$$\bullet \int_{-\infty}^{\infty} e^x dx$$

$$= \int_{-\infty}^0 e^x dx + \underbrace{\int_0^{\infty} e^x dx}$$

$$\lim_{b \rightarrow \infty} \int_0^b e^x dx = \lim_{b \rightarrow \infty} \left[e^x \right]_0^b = \lim_{b \rightarrow \infty} e^b - 1 = \infty$$

A: Convergent
B: Divergent
E: Neither

Application to Radioactive Decay



- Plutonium-238 has a half-life of 87.7 years.
- You have a sample of plutonium that releases 100 rem/yr.
- How much total rems of radioactivity is emitted by the sample over eternity?

Radioactive decay: $P(t) = P(0)e^{-kt}$

$$\frac{1}{2} = e^{-87.7k} \Rightarrow -87.7k = \ln \frac{1}{2}$$
$$k = 0.0079$$

REMs released: $\int_0^{\infty} P(t) \cdot r \, dt$, where r is a rate of rem release per amt of Plutonium. $P(0) \cdot r = 100$.

$$= \int_0^{\infty} P(0) \cdot r \cdot e^{-kt} \, dt = \int_0^{\infty} 100 e^{-kt} \, dt = \left[-\frac{1}{k} \cdot 100 e^{-kt} \right]_0^{\infty}$$
$$= \lim_{b \rightarrow \infty} \left[-\frac{100}{k} (-1 + e^{-bk}) \right] = \frac{100}{k} = 12658 \text{ rems}$$

Reminder: L'Hopital's Rule

- If $f(x)$ and $g(x)$ are differentiable on an open interval I except possibly at a point $c \in I$ (c is allowed to be $\pm\infty$), and if $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$ (except possibly at $g'(c)$) then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

- Lemma: If $r > 0$ and n is any number, then

$$\lim_{x \rightarrow \infty} x^n e^{-rx} = \lim_{x \rightarrow \infty} \frac{x^n}{e^{rx}} = 0$$

Ex: $\int_0^{\infty} x e^{-x} dx$

DETAIL

Solve antiderivative first

$$\int x e^{-x} dx$$

$$\int u dv = uv - \int v du$$

$$u = x$$

$$v = -e^{-x}$$

$$du = dx$$

$$dv = e^{-x} dx$$

$$= -x e^{-x} + \int (+e^{-x}) dx$$

$$= -x e^{-x} - e^{-x} + C$$

$$\begin{aligned} \Rightarrow \int_0^{\infty} x e^{-x} dx &= \lim_{b \rightarrow \infty} \left[-x e^{-x} - e^{-x} \right] \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left[\cancel{(-b e^{-b})} - \cancel{e^{-b}} \right] - \left[\underline{0} - 1 \right] \\ &= 1 \end{aligned}$$

Integration Review

- $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$, where $\Delta x = \frac{1}{b-a}$ and $x_i = a + i\Delta x$.
- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
- $\int_a^b f(x) dx + \int_a^b g(x) dx = \int_a^b [f(x) + g(x)] dx$
- If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x) dx$ is the area under the curve $y = f(x)$ in the interval $[a, b]$
- FTC: If f is continuous and $\int f(x) dx = F(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$. Also, $\frac{d}{dx} \int_a^x f(t) dt = f(x)$.
- U-substitution: $\int_{x=a}^{x=b} f'(u) \cdot u' dx = f(u) \Big|_{u=u(a)}^{u=u(b)}$
- Integration by parts (DETAIL): $\int u dv = uv - \int v du$.
- Volume = $\int_a^b A(x) dx$, and $A(x) = \pi(f(x))^2$ for a solid of revolution