Matrix operations
Lecture 3a: 2023-01-23

MAT A35 – Winter 2023 – UTSC

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How excited are you for your first quiz?

A. Super pumped
B. Good
C. Meh
D. Not looking forward
E. In a state of dread

Scalars to Vectors

- A scalar is a single number $a \in \mathbb{R}$.

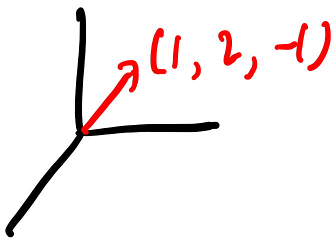
Ex. UTSC is 59 years old

- A vector is a collection of n numbers $v = \{v_1, v_2, \dots, v_n\} \in \mathbb{R}^n$.

Ex. UTSC's GPS coordinates are $(43.78, -78.18)$

Ex. A bird pop. has 100 hatchlings and 200 adults. Can rep. as pop. vector $(100, 200)$

Ex.

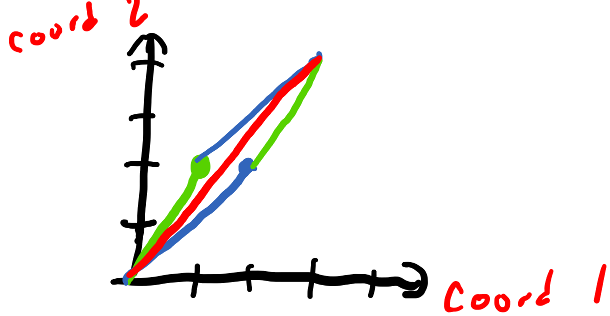


A pt or direction + magnitude in 3D

Operations on Vectors

- Addition: only works on same size vectors.

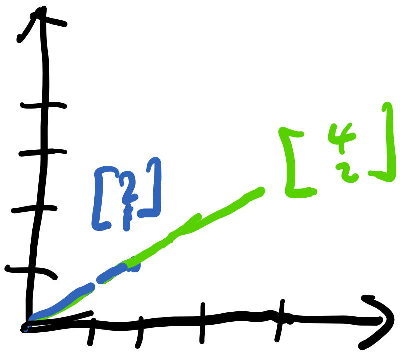
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$



Ex. $\underline{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} + \underline{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \underline{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$

- Multiplication by Scalar

$$a\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \end{bmatrix}$$



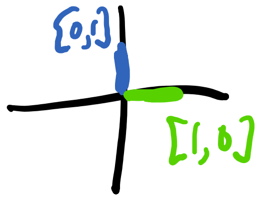
Ex. $2\underline{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \underline{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}$

Dot Product on Vectors

- Given two vectors of the same size $a, b \in \mathbb{R}^n$, where $a = \{a_1, \dots, a_n\}$ and $b = \{b_1, \dots, b_n\}$, the dot product $a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$, a scalar.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = -1 + 8 + 6 = 13$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 + 0 = 0$$



- The dot product intuitively tells you how similar two vectors are.

Try it out

- $\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}$ $= -2+2=0$

A: $\begin{pmatrix}-2\\2\end{pmatrix}$, B: 0, C: -3, D: $\begin{pmatrix}-1\\3\end{pmatrix}$, E: None

- $\begin{pmatrix}-2\\5\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}$

A: $\begin{pmatrix}-1\\7\end{pmatrix}$, B: $\begin{pmatrix}-1\\7\\3\end{pmatrix}$, C: 8, D: 0, E: None

- $0\begin{pmatrix}1\\2\\3\end{pmatrix}-2\begin{pmatrix}0\\0\\0\end{pmatrix}$

A: $\begin{pmatrix}0\\0\\0\end{pmatrix}$ B: $\begin{pmatrix}1\\2\\3\end{pmatrix}$, C: $\begin{pmatrix}-1\\0\\1\end{pmatrix}$, D: 0, E: None

$\begin{bmatrix}0\\0\\0\end{bmatrix}-\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Vectors to Matrices

- A matrix is a rectangular grid of scalars, with several important properties.
  - A matrix $A = [a_{ij}]$ with size $m \times n$ has m rows and n columns, and a_{ij} represents the entry in the ith row and jth column.
  - Like vectors, can add matrices, and multiply by a scalar.

$a_{3,2} = 6$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \boxed{6} \end{bmatrix}$ is a $\underline{3} \times \underline{2}$ matrix, $a_{21} = 3$

$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix}$

$A + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \\ 6 & 6 \end{bmatrix}$

Matrix transposition

- Let a matrix $A = [a_{ij}]$ with size $m \times n$ has m rows and n columns, where a_{ij} represents the entry in the ith row and jth column.
- Then the transpose $B = [b_{ij}] = A^T$ has size $n \times m$, which means it has n rows and m columns, and $b_{ij} = a_{ji}$.
- Transposition flips all terms across the diagonal line from the top-left to the bottom-right.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}^T}_{\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+1 & 0+3 \\ 2+0 & 1+4 \\ 5+1 & 6+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \\ 6 & 6 \end{bmatrix}$$

Row and column matrices = vectors

- A matrix with just 1 row is a row vector.

$[1 \quad 2 \quad 3]$ is a row vector

- A matrix with just 1 column is a column vector. Normally, when we say vector, we will refer to a column vector.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector

Matrix Products

- Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix. Then the product $C = AB$ is a $m \times p$ matrix such that
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ik}b_{kj}$$
- Alternately, can think of A as a collection of m stacked row vectors, and B as a collection of p column vectors. Then c_{ij} is the dot product of the ith row and the jth column, as vectors.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)\cdot(-3)+(2)(-1)+(3)(1) & (1)(-2)+(2)(0)+(3)(2) \\ 4\cdot -3+5\cdot -1+6\cdot 1 & 4\cdot -2+5\cdot 0+6\cdot 2 \end{bmatrix}$$

$\underline{2} \times \underline{3}$ $\quad$ $\underline{3} \times \underline{2}$

$$= \begin{bmatrix} -2 & 4 \\ -11 & 4 \end{bmatrix} \quad \underline{2} \times \underline{2}$$

Try it out

- $\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5 & 6\end{bmatrix}$

A: 32

B: $\begin{bmatrix}4\\10\\18\end{bmatrix}$

C: $\begin{bmatrix}4 & 10 & 18\end{bmatrix}$

D: $\begin{bmatrix}4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18\end{bmatrix}$

E: None

$3\times1 \quad 1\times3$

3×3

$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = 1\cdot4+2\cdot5+3\cdot6 = 32$

$1\times3 \quad 3\times1$

1×1

- $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}$

A: $\begin{bmatrix}17\\39\end{bmatrix}$

B: $\begin{bmatrix}5 & 10\\18 & 24\end{bmatrix}$

C: $\begin{bmatrix}23 & 34\end{bmatrix}$

D: 56

E: None

$2\times2 \quad 2\times1$

2×1

$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\\7\end{bmatrix}$

$2\times2 \quad 3\times1$

$\times$

$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}$

$= \begin{bmatrix}1\cdot5+2\cdot6\\3\cdot5+4\cdot6\end{bmatrix} = \begin{bmatrix}17\\39\end{bmatrix}$

Outer products

Inner product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\Longleftrightarrow \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 + a_2 b_2 + a_3 b_3 \end{bmatrix}$$

Outer product

$$\underbrace{\underset{3 \times 1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}}}_{} = \underset{3 \times 3}{\begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}}$$

Matrix multiplication using outer products

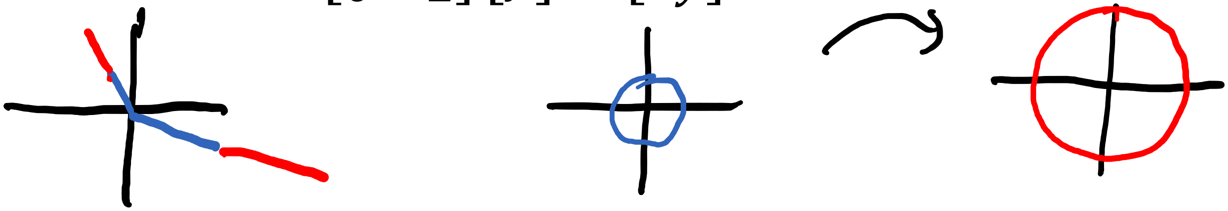
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} -3 & 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}\begin{bmatrix} -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 \\ -12 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ -5 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$$

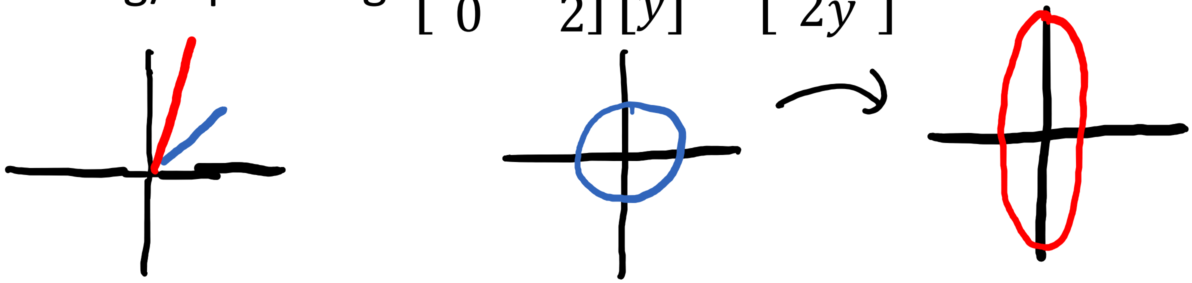
$$= \begin{bmatrix} -2 & 4 \\ -11 & 4 \end{bmatrix}$$

Matrices are transformations of vectors

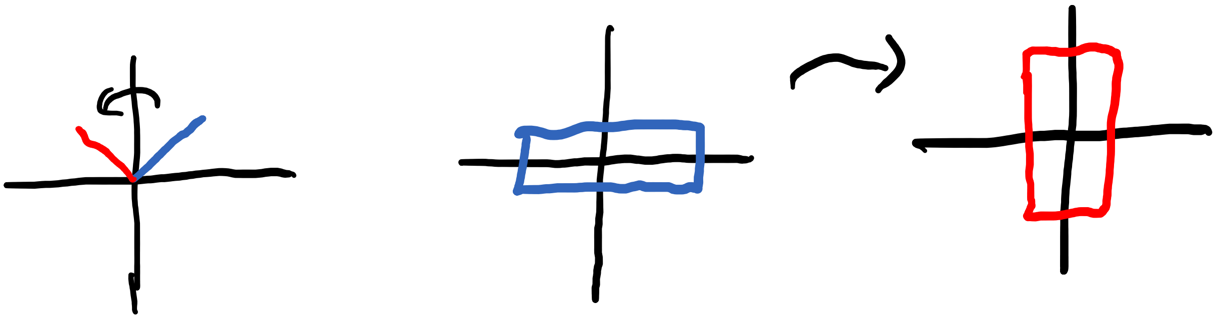
- Scaling operators: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$



- Stretching/squashing: $\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ 2y \end{bmatrix}$



- Rotations: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$



- Etc…

Application: Leslie Matrices

- Consider a rabbit population that can be divided into two age classes: young and adult.
  - Young rabbits can reach sexual maturity within 6-7 months.
  - Adult rabbits live on average for 9 years.
- Let's consider a *state vector* of the rabbit population:

$$\begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$$

- Each year, both the young and adult rabbits have a chance of surviving (survivability), and a number of offspring (fecundity), which we can encode in a *Leslie matrix*.

$$\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix}$$

Rabbit population - continued

$$\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \underline{\text{survivability of young}} & \underline{\text{survivability of adults}} \end{bmatrix} \begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of offspring of young} + \text{number of offspring of adults} \\ \text{number of surviving young} + \text{number of surviving adults} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of young rabbits the following year} \\ \text{number of adult rabbits the foollowing year} \end{bmatrix}$$

Ex

$$L = \overbrace{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}}^{L}$$

Year 1: young: adult: $\overset{x}{\begin{bmatrix} 100 \\ 10 \end{bmatrix}}$

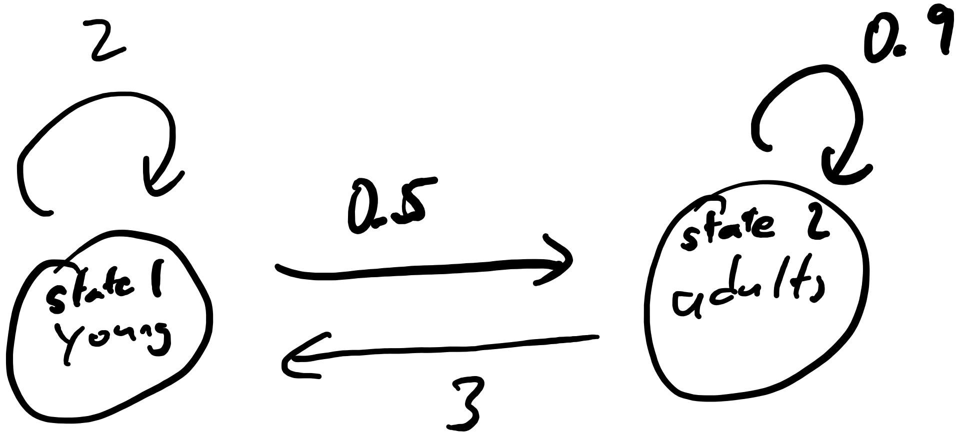
Year 2: $\overset{L}{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}} \overset{x}{\begin{bmatrix} 100 \\ 10 \end{bmatrix}} = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$

Year 3: $\overset{L}{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}} \overset{L}{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}} \overset{x}{\begin{bmatrix} 100 \\ 10 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}} \overset{Lx}{\begin{bmatrix} 230 \\ 59 \end{bmatrix}} = \overset{L^2x}{\begin{bmatrix} 637 \\ 168.1 \end{bmatrix}}$

Leslie Diagrams

- We can encode a Leslie matrix as a graph

$$L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$$

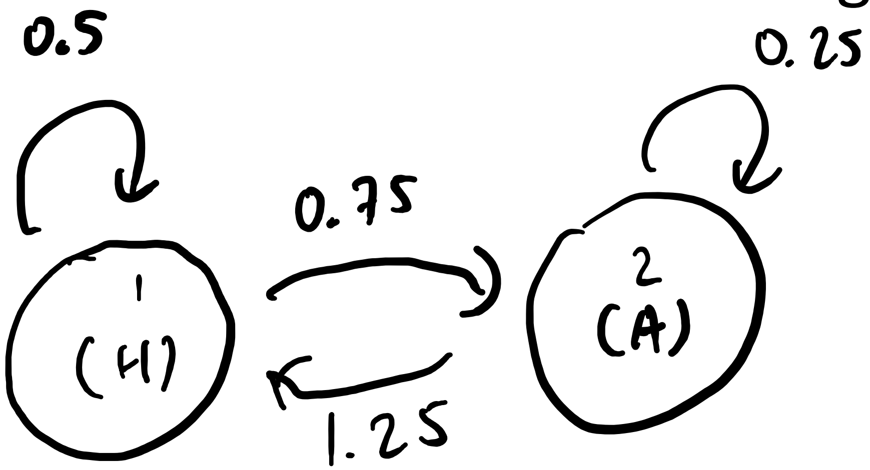


young next year = 2 · [young] + 3 · [adults]

adults next year = 0.5 · [young] + 0.9 · [adults]

Try it out

- A population of birds has the following Leslie diagram with 100 hatchlings (H) and 40 adults (A) in year 1. Estimate the number of hatchlings and young in year 2.



$$LX = \begin{bmatrix} 50 + 50 \\ 75 + 10 \end{bmatrix} = \begin{bmatrix} 100 \\ 85 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.5 & 1.25 \\ 0.75 & 0.25 \end{bmatrix} \qquad X = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$$

A: ~~100 hatchlings, 40 adults~~
B: 100 hatchlings, 85 adults
C: 200 hatchlings, 40 adults
D: 200 hatchlings, 85 adults
E: None

Matrix identity

- Identity matrix: a special $n \times n$ matrix $I_n = I$ such that $AI = A$ for any $m \times n$ matrix and $IA = A$ for any $n \times m$ matrix.

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$ has 1's on diagonal and 0's elsewhere.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Matrix algebra

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 17 \\ 39 \end{bmatrix} = \begin{bmatrix} 17 \\ -39 \end{bmatrix}$$

- $A(BC) = (AB)C$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\overbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ -39 \end{bmatrix}$$

- $A(B + C) = AB + AC$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

- $(B + C)A = BA + CA$

- $AB \neq BA$ (in general)

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Try it out

- $\underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right)$

A: $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$
B: $\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$
C: $\begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$
D: $\begin{bmatrix} 6 & 11 \\ 11 & 26 \end{bmatrix}$
E: None

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}\right)$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

A: $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$
B: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
C: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
D: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
E: None