Systems of linear equations
Lecture 3b: 2023-01-23

MAT A35 – Winter 2023 – UTSC

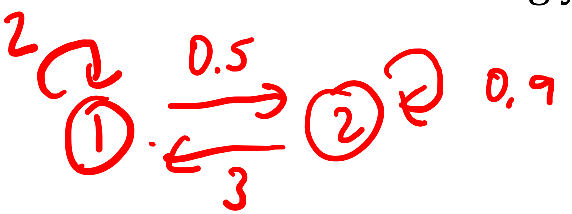
Prof. Yun William Yu

Rabbit population - reminder

$$\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix} \begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of offspring of young} + \text{number of offspring of adults} \\ \text{number of surviving young} + \text{number of surviving adults} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of young rabbits the following year} \\ \text{number of adult rabbits the following year} \end{bmatrix}$$



- Suppose you have a Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$ in Year 2. What was the population vector p_1 in Year 1?

Matrix Equation to System of Equations

- Suppose you have a Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $\mathrm{p}_2 = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$ in Year 2. What was the population vector p_1 in Year 1?

$$\underset{L}{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}} \underset{p_1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{p_2}{\begin{bmatrix} 230 \\ 59 \end{bmatrix}} \iff \underset{Lp_1}{\begin{bmatrix} 2x + 3y \\ 0.5x + 0.9y \end{bmatrix}} = \underset{p_2}{\begin{bmatrix} 230 \\ 59 \end{bmatrix}} \text{)}\times 4$$

$$\iff \begin{array}{rl} & 2x + 3y = 230 \\ -) & 2x + 3.6y = 236 \\ \hline & -0.6y = -6 \end{array}$$

$$\begin{cases} y = 10 \\ x = 100 \end{cases}$$

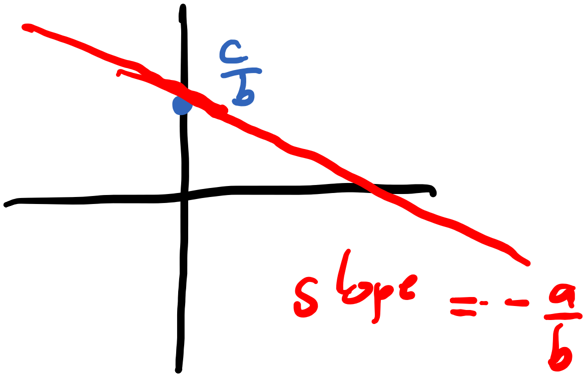
$$\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 10 \end{bmatrix} = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$$

Graphs of 2D linear equations

- Can visualize 2-variable equations as lines.

$$ax + by = c$$

$$\Rightarrow y = \frac{c - ax}{b} = \frac{c}{b} - \frac{a}{b}x$$



- Any point on the line is a solution to the equation.

$$x + 2y = 3$$

$$y = \frac{3}{2} - \frac{1}{2}x$$

$x = 0, \quad y = \frac{3}{2}$

$x = 1, \quad y = 1$

$x = 2, \quad y = \frac{1}{2}$

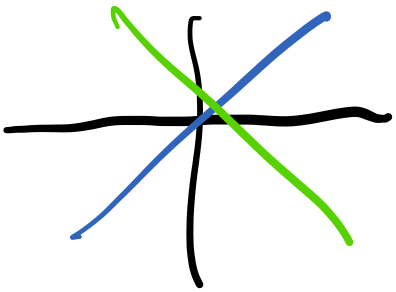
$\vdots$

Graphs of 2D linear systems

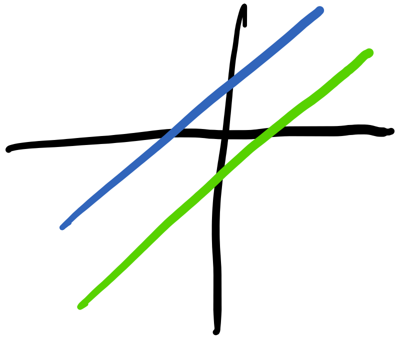
- A solution to a system of 2 linear equations with 2 variables has to be a solution to both of equations—I.e. it lies on both lines.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

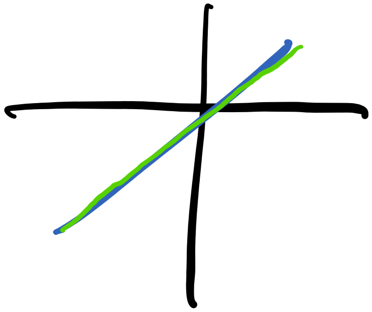
- Three possibilities for number of solutions



1 solution



No solutions



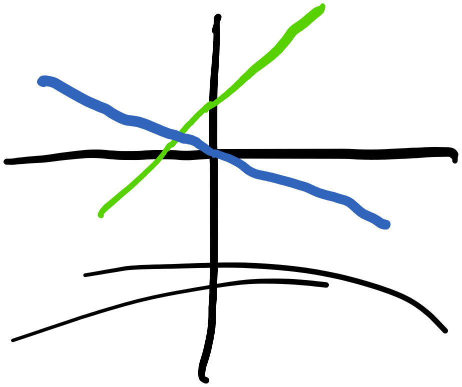
infinite solutions

(In)consistency and (in)dependence

- A system of equations is consistent if it has at least one solutions. Otherwise, it is inconsistent (no solutions).
- A system of equations is dependent if you can derive one of the equations from the other equations. Otherwise, the system is independent.
  - An equation that can be derived from the other equations is also called dependent, and one that cannot is called independent.

$y = -\frac{1}{2}x$

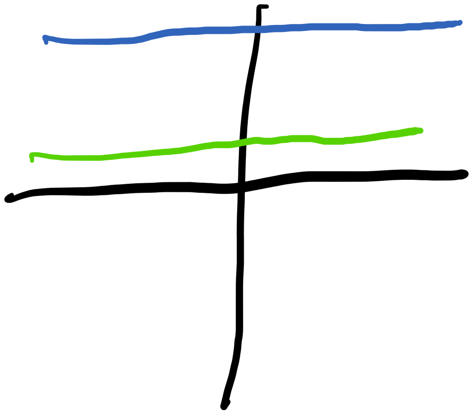
$y = x+1$



consistent & independent

$y = 4$

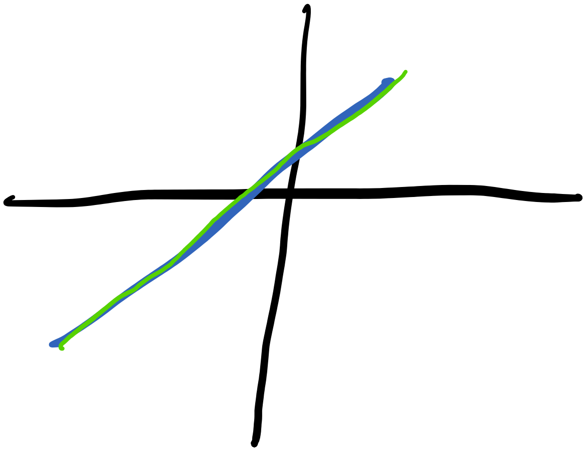
$y = 1$



inconsistent & independent

$2y = 2x+2$

$(y = x+1) \cdot 2$



consistent & dependent

Try it out

- $\begin{cases} x + 2y = 5 \\ x - 2y = 1 \end{cases}$

- $\begin{cases} x + 2y = 5 \\ x + 2y = 1 \end{cases}$

- $\begin{cases} x + 2y = 5 \\ -3x - 6y = -15 \end{cases}$

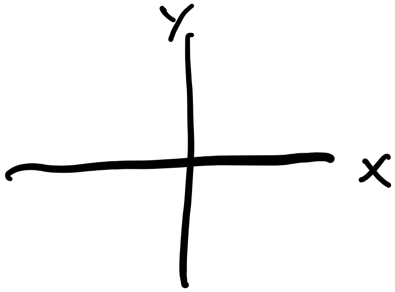
- $\begin{cases} (x + 2y + z = 5) \cdot 2 \\ 2x + 4y + 2z = 10 \\ x + 2y + z = 10 \end{cases}$

A: Consistent and independent
B: Inconsistent and independent
C: Consistent and dependent
D: Inconsistent and dependent
E: None

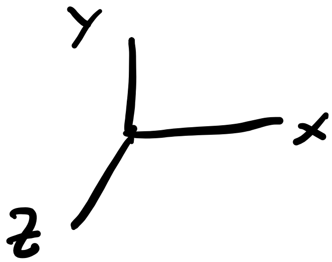
Properties of systems of equations

- Each variable in a system of equations can be thought of as a degree of freedom.

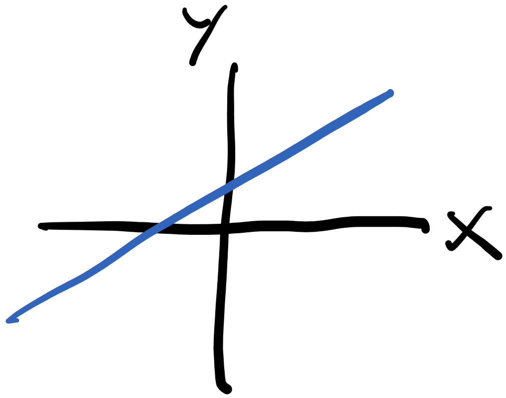
x, y

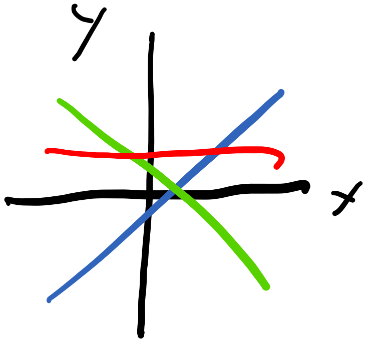


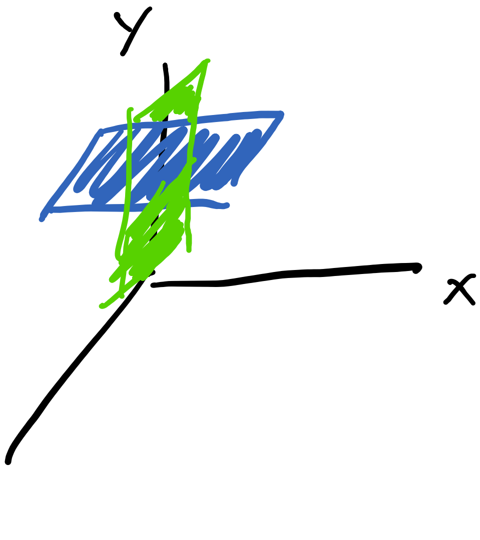
x, y, z



- Each independent equation constrains the system and removes a degree of freedom.







Properties of systems of equations

- A system of n linear equations with n variables has exactly 1 solution if and only if the system is independent and consistent.

$$\begin{cases} x+y=1 \\ x-y=5 \end{cases} \qquad \begin{aligned} 2x&=6 \\ x&=3 \end{aligned} \qquad y=-2$$

- If $m > n$, then a system of m linear equations with n variables does not have a solution if all the equations are independent.

$$\left.\begin{cases} x+y=1 \\ x-y=5 \end{cases}\right\} \; x=3,\; y=-2$$
$$2x+y=0 \;\Rightarrow\; 6-2=0 \quad \text{✗}$$

- If $m < n$, then a system of m linear equations with n variables has infinitely many solutions if the system is independent and consistent. (of course, a system with at least 2 equations can be inconsistent)

$$x+y=1 \quad ✓$$

$$\cancel{\begin{aligned} x+y+z&=1 \\ x+y+z&=0 \end{aligned}}$$

Substitution method

- Solve for a variable in one equation in terms of the other variables, and then substitute it into all the other equations.
- Iterate until you know the value of one variable.
- Then plug that variable value into all of the equations and repeat the entire process with one fewer variable.

Ex.
$$\begin{cases} 3x - 2y = 1 \\ -x + y = 1 \end{cases}$$

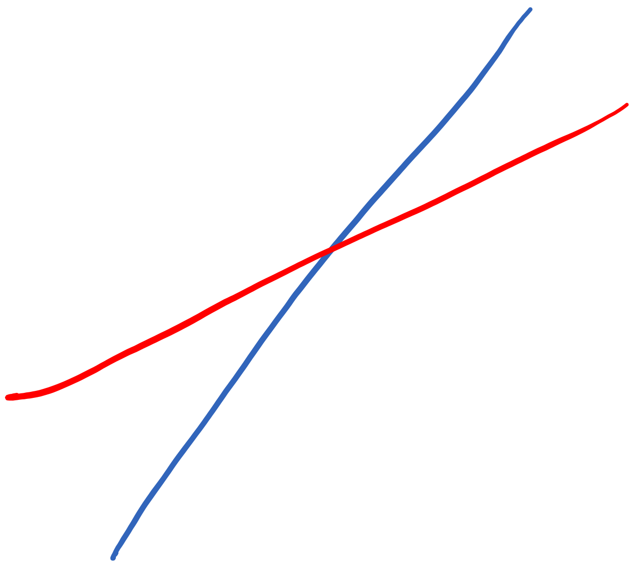
$$y = x + 1$$

$$3x - 2(x+1) = 1$$

$$3x - 2x - 2 = 1$$

$$x = 3$$

$$y = 4$$



Elimination Method

- Transform a system into an "equivalent" system with the same solutions using three types of operations:
  - Change (permute) the order of the equations.
  - Multiply an equation by a non-zero constant.
  - Add a multiple of one equation (A) to another (B). (B←cA+B)
- Goal is to eliminate variables
  - Can then use "back-substitution" to solve.
  - Can encode as an "augmented matrix"

$$\begin{cases} x+y=1 \\ x-y=0 \end{cases} \iff \begin{cases} x-y=0 \\ x+y=1 \end{cases}$$

$$\begin{cases} x+y=1 \\ x-y=0 \end{cases} \iff \begin{cases} 2x+2y=2 \\ x-y=0 \end{cases}$$

$$\begin{cases} x+y=1 \\ x-y=0 \end{cases} \iff \begin{cases} x+y=1 \\ 2x=1 \end{cases}$$

Ex.

$$x=\frac{1}{2}$$

$$\frac{1}{2}+y=1 \implies y=\frac{1}{2}$$

Example

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x-2y \\ -x+y \end{bmatrix}$$

Augmented matrix

$$\begin{cases} 3x - 2y = 1 \\ (-x + y = 1) \quad (-1) \end{cases} \qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

$R_2 \leftarrow -R_2$

$$\begin{cases} 3x - 2y = 1 \\ x - y = -1 \end{cases} \qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & -1 & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\begin{cases} x - y = -1 \\ 3x - 2y = 1 \end{cases} \ (-3) \qquad \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & -2 & 1 \end{array}\right] \qquad \begin{array}{cc|c} -3 & 3 & 3 \end{array} \quad +$$

$R_2 \leftarrow R_2 - 3R_1$

$$\begin{cases} x - y = -1 \\ y = 4 \end{cases} \ \text{back-sub} \qquad \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 4 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_2$

$$\begin{cases} x = 3 \\ y = 4 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

Elementary row operations

- Elementary row operations
  - Swap: Any row can be switched with any other row
  - Scale: Any row can be multiplied by a non-zero constant
  - Pivot: A multiple of one row can be added to another row
- If two matrices can be converted to one another via elementary row operations, then they are row-equivalent.

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

(Reduced) row-echelon form

- A matrix is in row-echelon form if:
  - If a row is not all 0's, then the first nonzero entry is a 1.
  - The leading 1 in a row is to the right of the leading one in the row above.
  - Every row with all 0's is at the bottom of the matrix
- A matrix is in reduced row-echelon form if in addition:
  - Each column containing a leading 1 in a row has all other entries 0.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 2y + 3z = 1$

$z = 0$

A: Row-echelon form
B: Reduced row-echelon form
C: Both A & B
E: None

Gauss-Jordan elimination

- Gaussian elimination is using elementary row operations to convert a matrix to row echelon form.
    - Work from left to right. Start by using swaps, scales, and pivots to convert the leftmost nonzero column to having a 1 as close to the top left as possible, and 0's everywhere else in the column.
    - Then iteratively repeat on the submatrix below and to the right of that 1. (i.e. freeze that row; don't do any more row operations to it)
    - All zero rows can be swapped to the bottom and ignored.
    - An all zero row except with a nonzero right augmented term means that the systems is inconsistent.
- Gauss-Jordan elimination is using elementary row operations to convert a matrix to reduced row echelon form.
    - Start with a Gaussian elimination to get to row echelon form.
    - For the bottom-right 1, use pivots to zero out the entries above it.
    - Iteratively repeat on the submatrix above and to the right of that 1 until you get all the way to the top.

Example

$$\begin{aligned} x \quad\quad\quad + z &= 2 \\ 2y + 2z &= 6 \\ -2x + 2y \quad\quad &= \cancel{2}\ 3 \end{aligned}$$

$$\begin{cases} x \quad + z = 2 \\ y + z = 3 \end{cases}$$

$0 = 1$

2 equations,

3 variables,

infinitely many solutions

inconsistent

no solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 6 \\ -2 & 2 & 0 & \cancel{2}\ 3 \end{array}\right]$$

2 0 2 | 4

$R_3 \leftarrow R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & 2 & 2 & \cancel{6}\ 7 \end{array}\right]$$

$R_2 \leftarrow \frac{1}{2} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & \cancel{6}\ 7 \end{array}\right]$$

0 2 2 | 6

$R_3 \leftarrow R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & \cancel{0}\ 1 \end{array}\right]$$

RREF

Example

Try it out

- Suppose you have a Leslie matrix $L = \begin{bmatrix} 1 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 160 \\ 68 \end{bmatrix}$ in Year 2, corresponding to 160 young, and 68 adults. How many adults were there in Year 1?

$$\begin{bmatrix} 1 & 3 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 160 \\ 68 \end{bmatrix}$$

$$x + 3y = 160$$

$$0.5x + 0.9y = 68$$

$$\left[\begin{array}{cc|c} 1 & 3 & 160 \\ 0.5 & 0.9 & 68 \end{array}\right] \xrightarrow{R_2 \leftarrow 2R_2} \left[\begin{array}{cc|c} 1 & 3 & 160 \\ 1 & 1.8 & 136 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$R_2 \leftarrow -1R_2$

$$\xrightarrow{R_2 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c} 1 & 3 & 160 \\ 0 & 1.2 & 24 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{R_2}{1.2}} \left[\begin{array}{cc|c} 1 & 3 & 160 \\ 0 & 1 & 20 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 100 \\ 0 & 1 & 20 \end{array}\right]$$

A: 100
B: 10
C: 160
D: 20
E: None