Systems of linear equations
Lecture 3b: 2023-01-23

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Rabbit population - reminder

$$\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix} \begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of offspring of young} + \text{number of offspring of adults} \\ \text{number of surviving young} + \text{number of surviving adults} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of young rabbits the following year} \\ \text{number of adult rabbits the following year} \end{bmatrix}$$

- Suppose you have a Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$ in Year 2. What was the population vector p_1 in Year 1?

Matrix Equation to System of Equations

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Graphs of 2D linear equations

- Can visualize 2-variable equations as lines.

- Any point on the line is a solution to the equation.

Graphs of 2D linear systems

- A solution to a system of 2 linear equations with 2 variables has to be a solution to both of equations—I.e. it lies on both lines.

- Three possibilities for number of solutions

(In)consistency and (in)dependence

- A system of equations is consistent if it has at least one solutions. Otherwise, it is inconsistent (no solutions).
- A system of equations is dependent if you can derive one of the equations from the other equations. Otherwise, the system is independent.
  - An equation that can be derived from the other equations is also called dependent, and one that cannot is called independent.

Try it out

- $\begin{cases} x + 2y = 5 \\ x - 2y = 1 \end{cases}$

- $\begin{cases} x + 2y = 5 \\ x + 2y = 1 \end{cases}$

- $\begin{cases} x + 2y = 5 \\ -3x - 6y = -15 \end{cases}$

- $\begin{cases} x + 2y + z = 5 \\ 2x + 4y + 2z = 10 \\ x + 2y + z = 10 \end{cases}$

A: Consistent and independent
B: Inconsistent and independent
C: Consistent and dependent
D: Inconsistent and dependent
E: None

Properties of systems of equations

- Each variable in a system of equations can be thought of as a degree of freedom.

- Each independent equation constrains the system and removes a degree of freedom.

Properties of systems of equations

- A system of n linear equations with n variables has exactly 1 solution if and only if the system is independent and consistent.

- If $m > n$, then a system of m linear equations with n variables does not have a solution if all the equations are independent.

- If $m < n$, then a system of m linear equations with n variables has infinitely many solutions if the system is independent and consistent. (of course, a system with at least 2 equations can be inconsistent)

Substitution method

- Solve for a variable in one equation in terms of the other variables, and then substitute it into all the other equations.
- Iterate until you know the value of one variable.
- Then plug that variable value into all of the equations and repeat the entire process with one fewer variable.

Elimination Method

- Transform a system into an "equivalent" system with the same solutions using three types of operations:
  - Change (permute) the order of the equations.
  - Multiply an equation by a non-zero constant.
  - Add a multiple of one equation (A) to another (B). (B←cA+B)
- Goal is to eliminate variables
  - Can then use "back-substitution" to solve.
  - Can encode as an "augmented matrix"

Example

Elementary row operations

- Elementary row operations
  - Swap: Any row can be switched with any other row
  - Scale: Any row can be multiplied by a non-zero constant
  - Pivot: A multiple of one row can be added to another row
- If two matrices can be converted to one another via elementary row operations, then they are row-equivalent.

(Reduced) row-echelon form

- A matrix is in row-echelon form if:
  - If a row is not all 0's, then the first nonzero entry is a 1.
  - The leading 1 in a row is to the right of the leading one in the row above.
  - Every row with all 0's is at the bottom of the matrix
- A matrix is in reduced row-echelon form if in addition:
  - Each column containing a leading 1 in a row has all other entries 0.

A: Row-echelon form
B: Reduced row-echelon form
C: Both A & B
E: None

Gauss-Jordan elimination

- Gaussian elimination is using elementary row operations to convert a matrix to row echelon form.
  - Work from left to right. Start by using swaps, scales, and pivots to convert the leftmost nonzero column to having a 1 as close to the top left as possible, and 0's everywhere else in the column.
  - Then iteratively repeat on the submatrix below and to the right of that 1. (i.e. freeze that row; don't do any more row operations to it)
  - All zero rows can be swapped to the bottom and ignored.
  - An all zero row except with a nonzero right augmented term means that the systems is inconsistent.
- Gauss-Jordan elimination is using elementary row operations to convert a matrix to reduced row echelon form.
  - Start with a Gaussian elimination to get to row echelon form.
  - For the bottom-right 1, use pivots to zero out the entries above it.
  - Iteratively repeat on the submatrix above and to the right of that 1 until you get all the way to the top.

Example

Example

Try it out

- Suppose you have a Leslie matrix $L = \begin{bmatrix} 1 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 160 \\ 68 \end{bmatrix}$ in Year 2, corresponding to 160 young, and 68 adults. How many adults were there in Year 1?

A: 100
B: 10
C: 160
D: 20
E: None