Matrix inverses and determinants Lecture 4a: 2023-01-30

MAT A35 – Winter 2023 – UTSC

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How did you get here this morning?

A. Walked
B. Biked
C. Bus
D. Subway (+ Bus) (or GO train/bus)
E. Drove

"Dividing" by a matrix

Addition: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Subtraction: $\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Multiplication: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2+12 & 4+16 \\ 6+24 & 12+32 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 36 & 44 \end{bmatrix}$

Division?: $\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} \div \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

Inverses of multiplication = division

- One way to think about division in real numbers is multiplication by an inverse. Can we do something similar for matrices?

Ex. $15 \div 3 = 15 \cdot 3^{-1} = 15 \cdot \frac{1}{3} = \frac{15}{3} = 5$

Ex. $\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -14+15 & 7-5 \\ -30+33 & 15-11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}^{-1}$$

Multiplicative inverses for real numbers

- Let x be a real number. The *(multiplicative) inverse* of x is another real number $x^{-1} = \frac{1}{x}$ such that $xx^{-1} = x^{-1}x = 1$.

$1^{-1} = 1$ $\quad 2^{-1} = \frac{1}{2}$ $\quad 3^{-1} = \frac{1}{3}$ $\quad \pi^{-1} = \frac{1}{\pi}$ $\quad$ (0^{-1} = not a number)

- Reversal of multiplication: $x^{-1}(xy) = (x^{-1}x)y = 1 \cdot y = y$

$\frac{1}{2}(2 \cdot 3) = \left(\frac{1}{2} \cdot 2\right) \cdot 3 = 3$

$\shortparallel$

$\frac{1}{2} \cdot 6 = 3$

$y \cdot 0 = 0$

cannot be reversed

Matrix inverses (for square matrices)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Let A be a square matrix. The *(multiplicative) inverse* of A is a matrix A^{-1} with the property that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.
  - If A has an inverse, then it is *invertible* or *nonsingular*.
  - If A does not have an inverse, then it is *noninvertible* or *singular.*
  - Theorem: for a square matrix, if $AA^{-1} = I$, then $A^{-1}A = I$.

Ex.

$$\begin{bmatrix} -1 & \frac{1}{2} \\ 3/4 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -2+3 & -4+4 \\ 3/2 - 3/2 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ 3/4 & -\frac{1}{4} \end{bmatrix}$$

Finding a matrix inverse

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2x+4y & 2z+4w \\ 6x+8y & 6z+8w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} 2x+4y=1 \\ 6x+8y=0 \end{cases} \qquad \begin{cases} 2z+4w=0 \\ 6z+8w=1 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 1 \\ 6 & 8 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 6 & 8 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right]$$

Can combine both augmented systems

Finding a matrix inverse (cont.)

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right]$$

$R_1 \leftarrow \frac{1}{2} R_1$

$R_2 \leftarrow \frac{1}{6} R_2$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & 4/3 & 0 & \frac{1}{6} \end{array}\right]$$

$R_2 \leftarrow R_1 - R_2$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 2/3 & \frac{1}{2} & -\frac{1}{6} \end{array}\right]$$

$R_2 \leftarrow \frac{3}{2} \cdot R_2$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & 3/4 & -\frac{1}{4} \end{array}\right]$$

$R_1 \leftarrow R_1 - 2R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 3/4 & -\frac{1}{4} \end{array}\right]$$

$$\begin{bmatrix} x & w \\ y & z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3/4 & -\frac{1}{4} \end{bmatrix}$$

Matrix inversion through Gauss-Jordan

- Let A be a square $n \times n$ matrix. If we can row reduce the augmented matrix $[A|I]$ to the form $[I|B]$, then $A^{-1} = B$. Otherwise, the matrix A does not have an inverse.

Ex.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

no inverse

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}^{-1} \text{ does not exist}$$

Try it out

- Remember the Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ from our rabbit population model. Find the multiplicative inverse of L.

Solve $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0.5 & 0.9 & 0 & 1 \end{array}\right]$

$R_1 \leftarrow \frac{1}{2} R_1$

$R_2 \leftarrow 2 R_2$

$\left[\begin{array}{cc|cc} 1 & 3/2 & \frac{1}{2} & 0 \\ 1 & 1.8 & 0 & 2 \end{array}\right]$

$R_2 \leftarrow R_2 - R_1$

$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 0.3 & -\frac{1}{2} & 2 \end{array}\right]$

$R_2 \leftarrow \frac{10}{3} \cdot R_2$

$\left[\begin{array}{cc|cc} 1 & 3/2 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{5}{3} & \frac{20}{3} \end{array}\right]$

$R_1 \leftarrow R_1 - \frac{3}{2} R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -10 \\ 0 & 1 & -\frac{5}{3} & \frac{20}{3} \end{array}\right]$

A: $\begin{bmatrix} -2 & -3 \\ -0.5 & -0.9 \end{bmatrix}$

B: $\begin{bmatrix} 3 & -10 \\ -\frac{5}{3} & \frac{20}{3} \end{bmatrix}$

C: $\begin{bmatrix} 2 & 0.5 \\ 0.9 & 1 \end{bmatrix}$

D: $\begin{bmatrix} 3 & -\frac{5}{3} \\ -10 & \frac{20}{3} \end{bmatrix}$

E: None

Solving linear systems using inverses

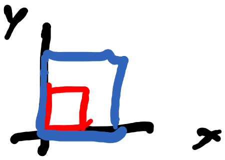
- Suppose $Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b$, where x is an unknown vector. Then we can solve $Ax = b$ by multiplying both sides on the *left* with A^{-1} if it exists. $x = A^{-1}Ax = A^{-1}b$
- Suppose you have a Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$ in Year 2. What was the population vector p_1 in Year 1?

$$p_1 = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}^{-1} \begin{bmatrix} 230 \\ 59 \end{bmatrix} = \begin{bmatrix} 3 & -10 \\ -\frac{5}{3} & \frac{20}{3} \end{bmatrix} \begin{bmatrix} 230 \\ 59 \end{bmatrix} = \begin{bmatrix} 690 - 590 \\ -\frac{1150}{3} + \frac{1180}{3} \end{bmatrix} = \begin{bmatrix} 100 \\ 10 \end{bmatrix}$$

When does a matrix have an inverse?

- Recall that matrices are transformations of vectors.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

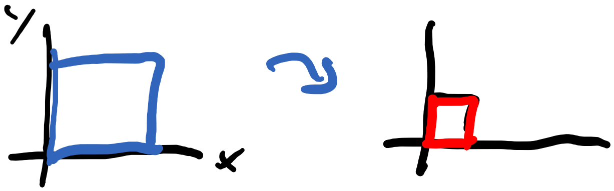


$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- A matrix has an inverse when you can reverse the transformation.

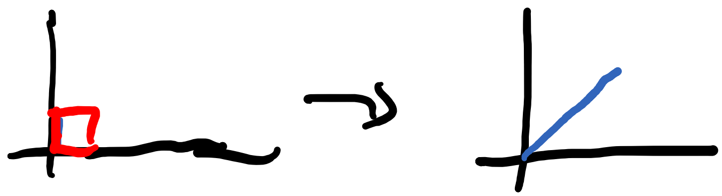
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$



- But if a matrix sends two points to the same point, then you can't reverse that mapping.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

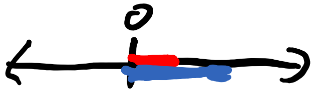
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$$



Matrices and length/area/volume scaling

- When a matrix squashes 1D line to a 0D point, that's irreversible.
  - Note that the length of a line gets scaled, but you get 0 length for a point.

$[2][x] = [2x]$



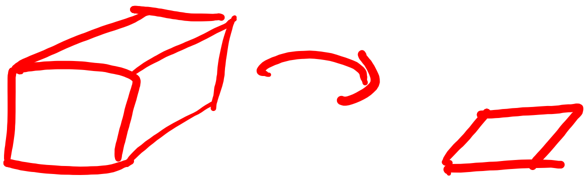
$[0][x] = [0]$

- When a matrix squashes a 2D square to a 1D line, that's irreversible.
  - Note that the area of a square gets scaled, but a line has area 0.

Previous page

- When a matrix squashes a 3D cube to a 2D plane, that's irreversible.
  - Note that a cube has nonzero volume, but a flat shape has volume 0.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



Matrix Determinants

- The determinant of a 1×1 matrix $[a]$ is a.
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - Note that even though the notation $|\ \ |$ looks like absolute values, determinants can be positive or negative.

Ex.

$$|[-1]| = -1$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \underline{1 \cdot 4} - \underline{2 \cdot 3} = -2$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$$

Try it out (find determinant)

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- $\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = 2$

A: 0
B: 1
C: 2
D: 3
E: None

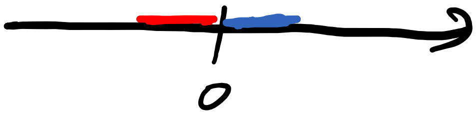
- $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= 1 - 1 = 0$

A: 0
B: 1
C: 2
D: 3
E: None

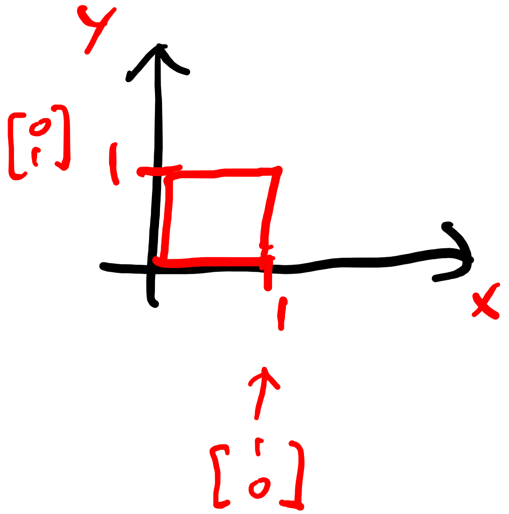
Determinants = (signed) scaling factor

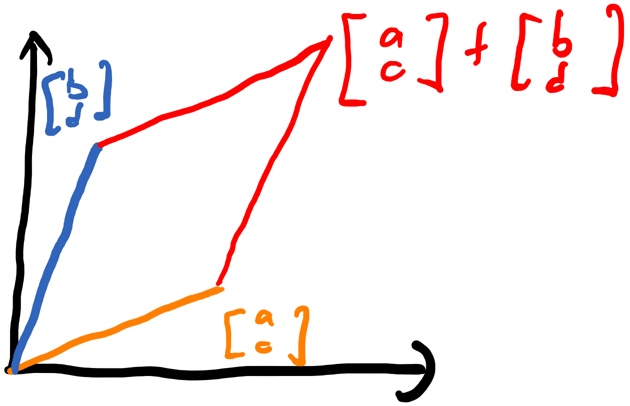
1D $\quad [2][x] = [2x] \qquad\qquad [-1][x] = [-x]$



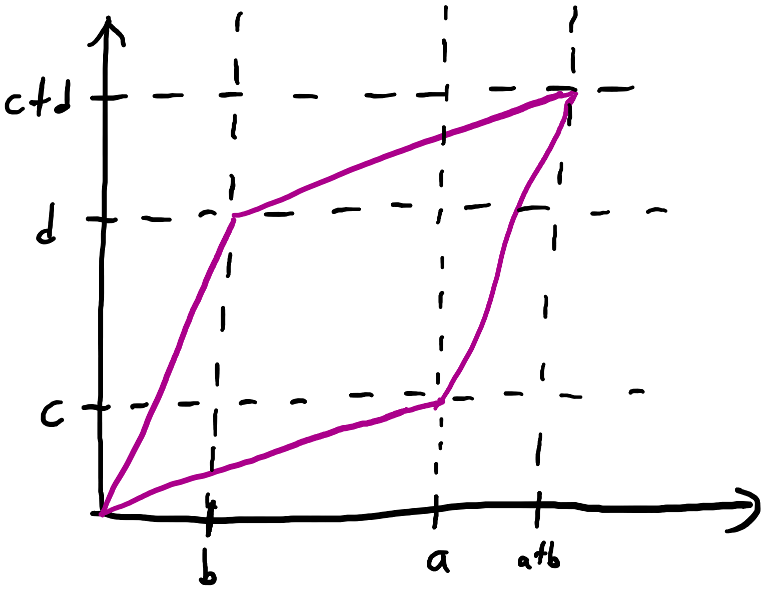
2D $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ ← interpretation?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



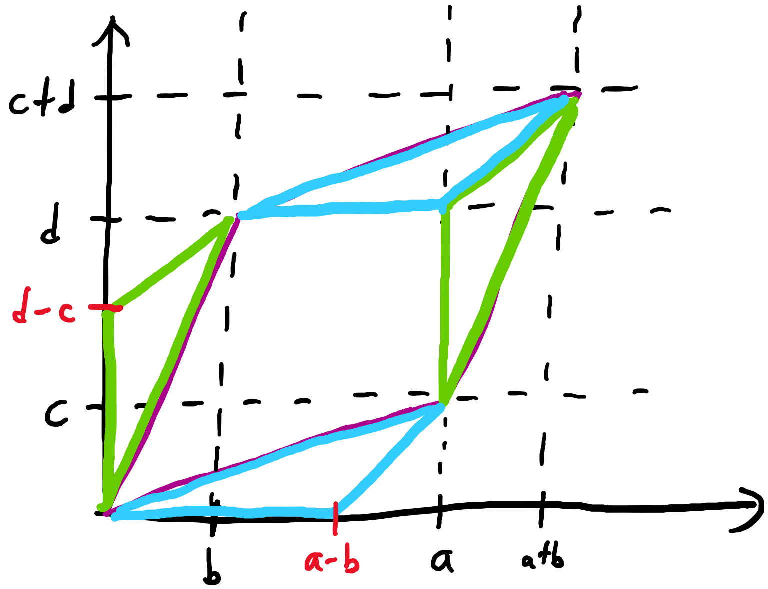


Area of parallelogram



Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



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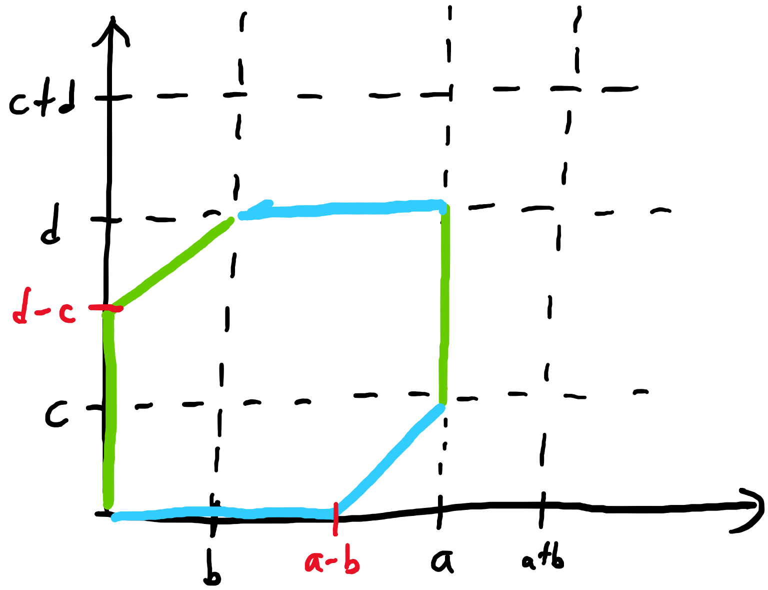
Area of parallelogram



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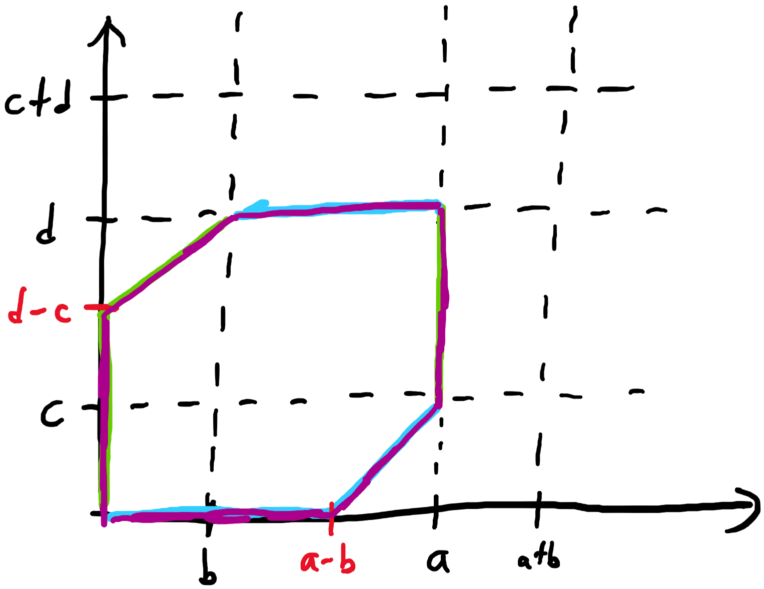
Area of parallelogram



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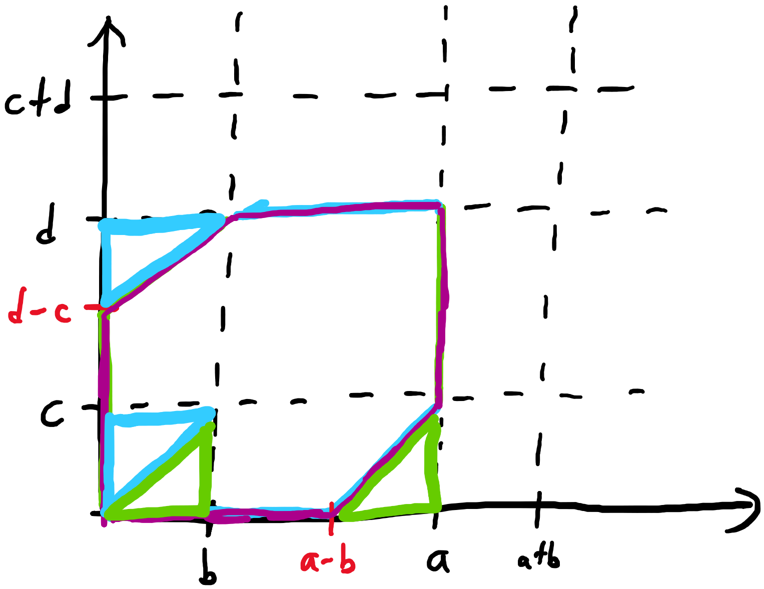
Area of parallelogram



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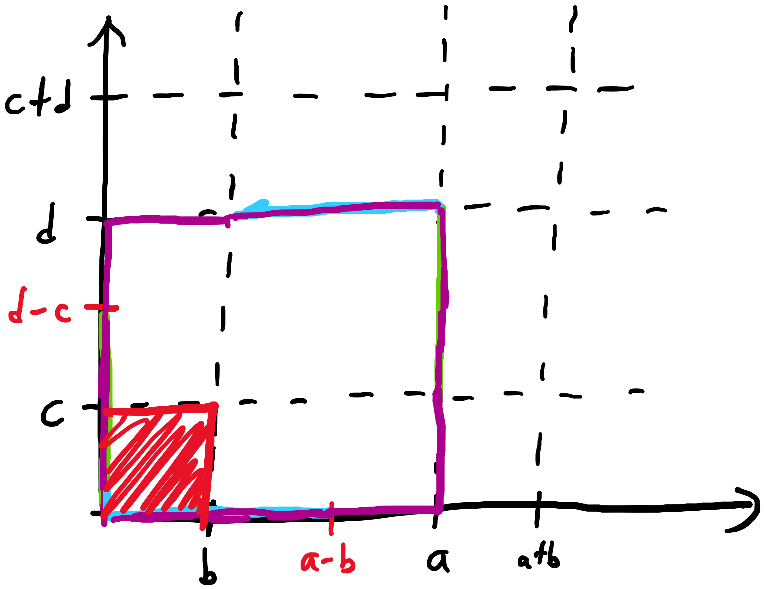
Area of parallelogram



Area =
$ad - bc$

Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Determinants and invertibility

- A square matrix is invertible if and only if its determinant is nonzero.
  - i.e. If a matrix squashes away a dimension, then it is not invertible, and vice versa.

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$ so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible

- If A is a square matrix, and $Ax = 0$ for some vector $x \neq 0$, then $\det A = 0$.
  - i.e. If a matrix squashes some nonzero vector to zero, then it is not invertible.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible

Determinants and matrix multiplication

- Since matrices are transformations, and determinants are a signed area, you can multiply together determinants:
- $\det(AB) = \det(A)\det(B)$, assuming A and B are square matrices of the same size.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 4 \qquad \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6 - 2 = 4$$

Determinants, minors, and cofactors

- Let $A = [a_{ij}]$ be a square $n \times n$ matrix. Then we can define the ijth *minor* M_{ij} of A as the determinant of the matrix where you have removed the ith row and the jth column of A.
- The ijth cofactor C_{ij} of A is $C_{ij} = (-1)^{i+j} M_{ij}$.
- The determinant of A can be defined recursively by $|A| = a_{11}C_{11} + \cdots a_{1n}C_{1n}$ the sum of the entries in the first row and their respective cofactors.
  - (you can expand along any row or column using this formula)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{aligned} C_{11} &= M_{11} \\ C_{12} &= -M_{12} \end{aligned}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

3x3 determinant memory aid

$$\left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

− for upward diag

+ for downward diag

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Example

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 6 & 5 & 0 \end{vmatrix} = 0\cdot\begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} - 1\cdot\begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} + 0\cdot\begin{vmatrix} 2 & 1 \\ 6 & 5 \end{vmatrix}$$

$$= 0 - 1\cdot(2\cdot 0 - 3\cdot 6) + 0 = 18$$

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 2 & 1 & 3 & 2 \\ 6 & 5 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -0\begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 5 & 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 & 1 \\ 2 & 3 & 2 \\ 6 & 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 6 & 5 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 6 & 5 & 0 \end{vmatrix}$$

$$= 2\cdot 18 = 36$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Try it out

- $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$$= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45-48) - 2(36-42) + 3 \cdot (32-35)$$

$$= -3 + 12 - 9 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A: -1
B: 0
C: 1
D: 2
E: None

- $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

$$= 2 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= 2 \cdot 2 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2 \cdot 2 \cdot 2 \cdot \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^5 = 32$$

A: 2
B: 5
C: 10
D: 32
E: None