Matrix inverses and determinants
Lecture 4a: 2023-01-30

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

"Dividing" by a matrix

Addition: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Subtraction: $\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Multiplication: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2+12 & 4+16 \\ 6+24 & 12+32 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 36 & 44 \end{bmatrix}$

Division?: $\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} \div \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

Inverses of multiplication = division

- One way to think about division in real numbers is multiplication by an inverse. Can we do something similar for matrices?

Ex. $15 \div 3 = 15 \cdot 3^{-1} = 15 \cdot \frac{1}{3} = \frac{15}{3} = 5$

Ex. $\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -14+15 & 7-5 \\ -30+33 & 15-11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}^{-1}$$

Multiplicative inverses for real numbers

- Let x be a real number. The *(multiplicative) inverse* of x is another real number $x^{-1} = \frac{1}{x}$ such that $xx^{-1} = x^{-1}x = 1$.

- Reversal of multiplication: $x^{-1}(xy) = (x^{-1}x)y = 1 \cdot y = y$

Matrix inverses (for square matrices)

- Let A be a square matrix. The *(multiplicative) inverse* of A is a matrix A^{-1} with the property that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.
  - If A has an inverse, then it is *invertible* or *nonsingular*.
  - If A does not have an inverse, then it is *noninvertible* or *singular.*
  - Theorem: for a square matrix, if $AA^{-1} = I$, then $A^{-1}A = I$.

Finding a matrix inverse

Finding a matrix inverse (cont.)

Matrix inversion through Gauss-Jordan

- Let A be a square $n \times n$ matrix. If we can row reduce the augmented matrix $[A|I]$ to the form $[I|B]$, then $A^{-1} = B$. Otherwise, the matrix A does not have an inverse.

Try it out

- Remember the Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ from our rabbit population model. Find the multiplicative inverse of L.

A: $\begin{bmatrix} -2 & -3 \\ -0.5 & -0.9 \end{bmatrix}$

B: $\begin{bmatrix} 3 & -10 \\ -\frac{5}{3} & \frac{20}{3} \end{bmatrix}$

C: $\begin{bmatrix} 2 & 0.5 \\ 0.9 & 1 \end{bmatrix}$

D: $\begin{bmatrix} 3 & -\frac{5}{3} \\ -10 & \frac{20}{3} \end{bmatrix}$

E: None

Solving linear systems using inverses

- Suppose $Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b$, where x is an unknown vector. Then we can solve $Ax = b$ by multiplying both sides on the *left* with A^{-1} if it exists. $x = A^{-1}Ax = A^{-1}b$
- Suppose you have a Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$ in Year 2. What was the population vector p_1 in Year 1?

When does a matrix have an inverse?

- Recall that matrices are transformations of vectors.

- A matrix has an inverse when you can reverse the transformation.

- But if a matrix sends two points to the same point, then you can't reverse that mapping.

Matrices and length/area/volume scaling

- When a matrix squashes 1D line to a 0D point, that's irreversible.
  - Note that the length of a line gets scaled, but you get 0 length for a point.
- When a matrix squashes a 2D square to a 1D line, that's irreversible.
  - Note that the area of a square gets scaled, but a line has area 0.
- When a matrix squashes a 3D cube to a 2D plane, that's irreversible.
  - Note that a cube has nonzero volume, but a flat shape has volume 0.

Matrix Determinants

- The determinant of a 1×1 matrix $[a]$ is a.
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - Note that even though the notation $|\ |$ looks like absolute values, determinants can be positive or negative.

Try it out (find determinant)

- $\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$

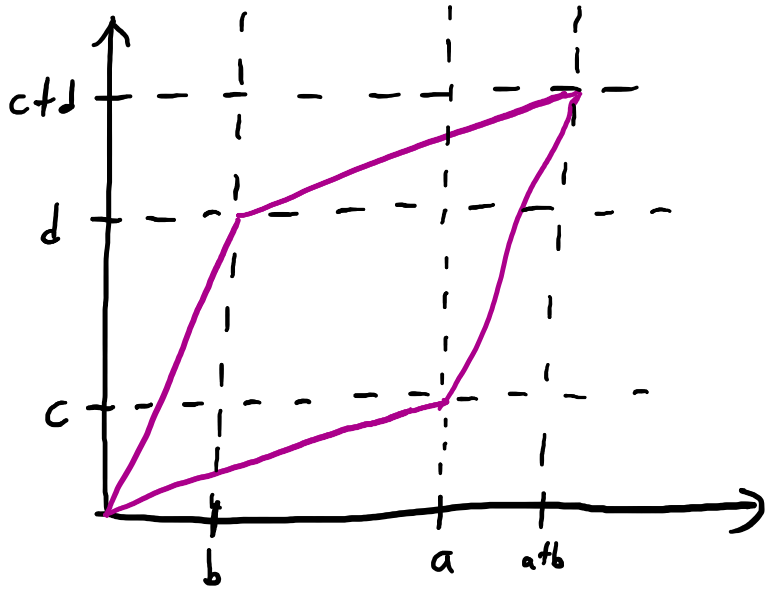
A: 0
B: 1
C: 2
D: 3
E: None

- $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

A: 0
B: 1
C: 2
D: 3
E: None

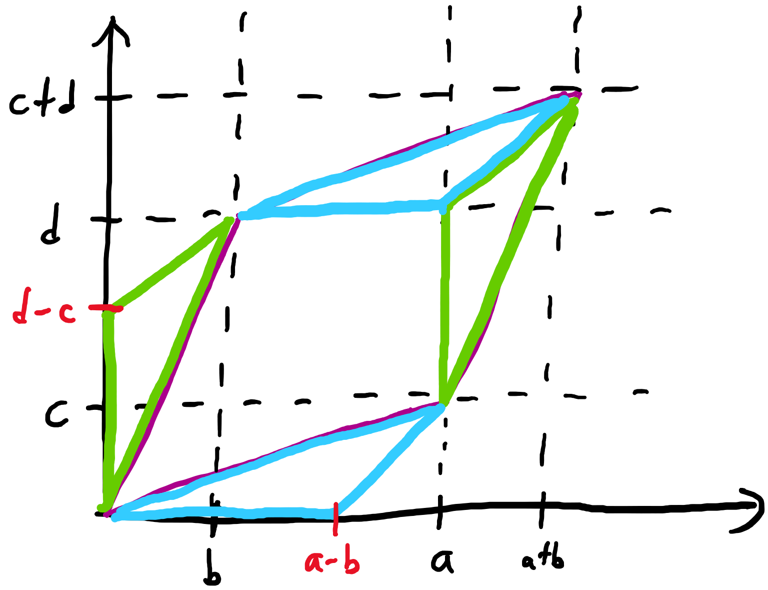
Determinants = (signed) scaling factor

Area of parallelogram



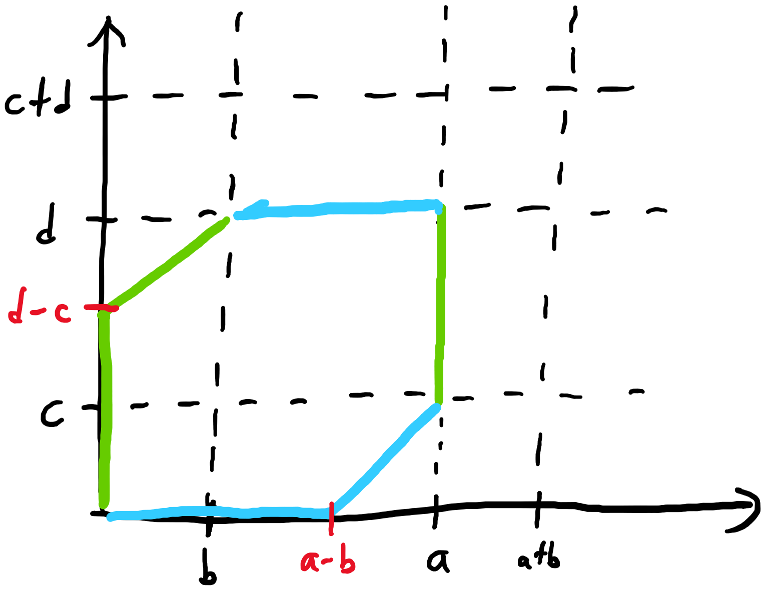
Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



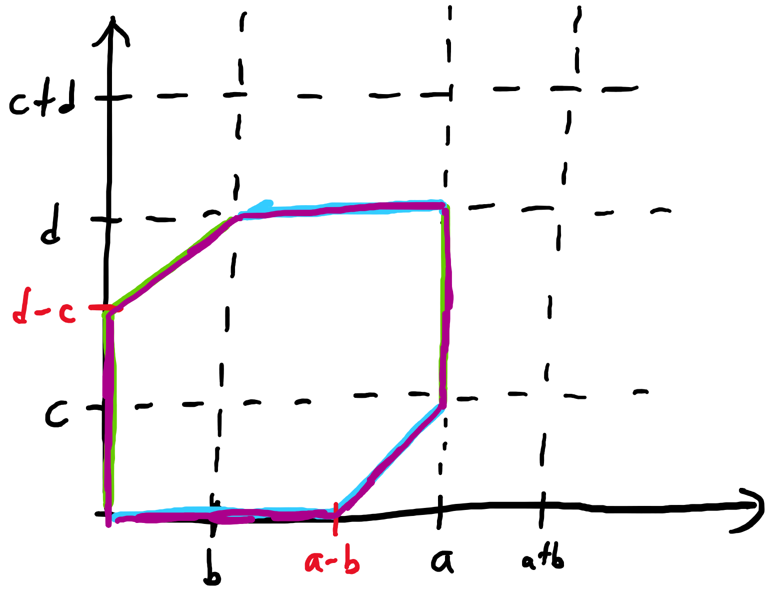
Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



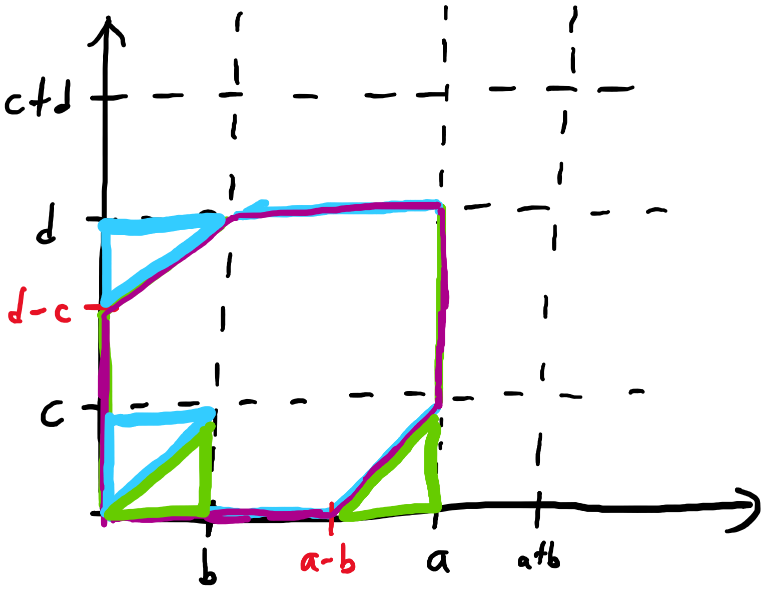
Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



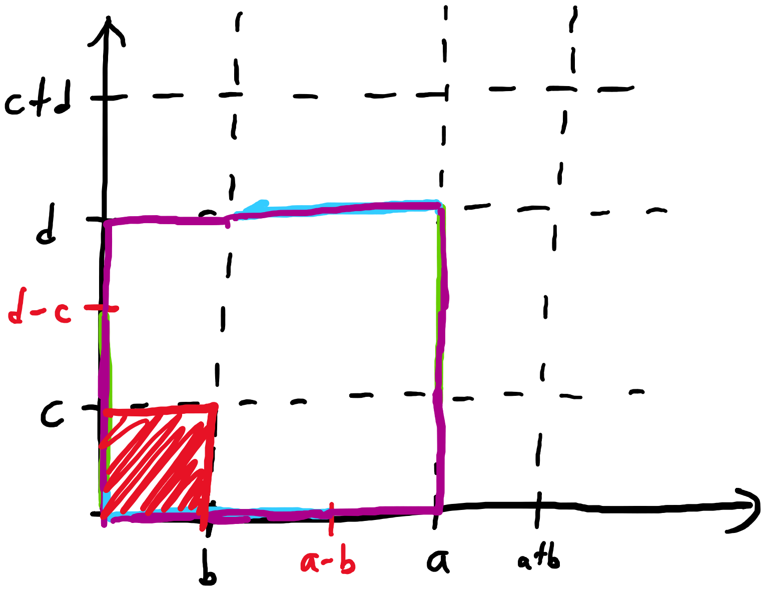
Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



Area =
ad - bc

Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Determinants and invertibility

- A square matrix is invertible if and only if its determinant is nonzero.
  - i.e. If a matrix squashes away a dimension, then it is not invertible, and vice versa.

- If A is a square matrix, and $Ax = 0$ for some vector $x \neq 0$, then $\det A = 0$.
  - i.e. If a matrix squashes some nonzero vector to zero, then it is not invertible.

Determinants and matrix multiplication

- Since matrices are transformations, and determinants are a signed area, you can multiply together determinants:
- $\det(AB) = \det(A)\det(B)$, assuming A and B are square matrices of the same size.

Determinants, minors, and cofactors

- Let $A = [a_{ij}]$ be a square $n \times n$ matrix. Then we can define the ijth *minor* M_{ij} of A as the determinant of the matrix where you have removed the ith row and the jth column of A.
- The ijth cofactor C_{ij} of A is $C_{ij} = (-1)^{i+j} M_{ij}$.
- The determinant of A can be defined recursively by $|A| = a_{11}C_{11} + \cdots a_{1n}C_{1n}$ the sum of the entries in the first row and their respective cofactors.
  - (you can expand along any row or column using this formula)

3x3 determinant memory aid

Example

Try it out

- $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

A: -1
B: 0
C: 1
D: 2
E: None

- $$\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

A: 2
B: 5
C: 10
D: 32
E: None