What is a basis?
Lecture 4b: 2023-01-30

MAT A35 – Winter 2023 – UTSC

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IC 343

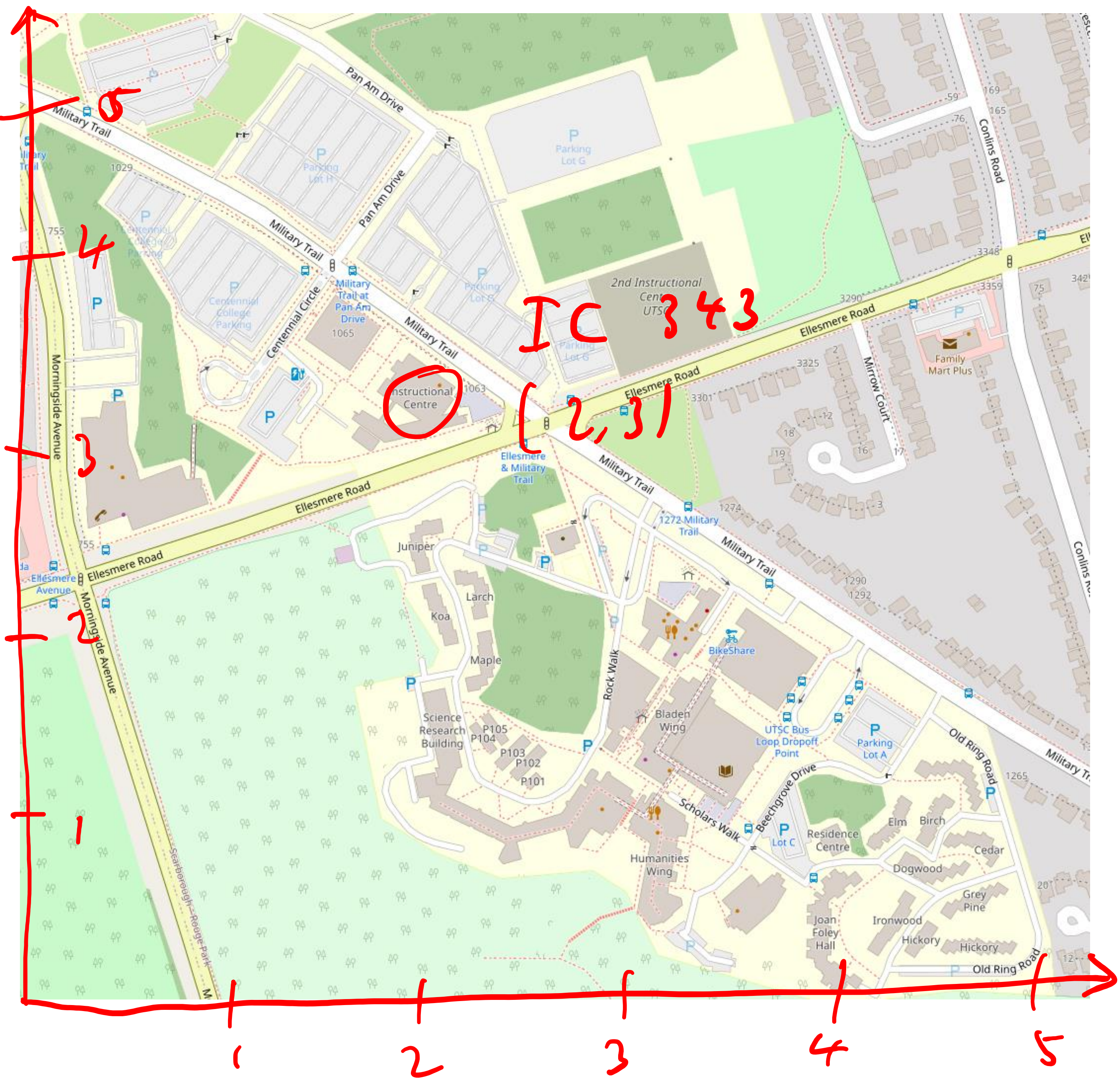
map grid

C2

(3, 4)

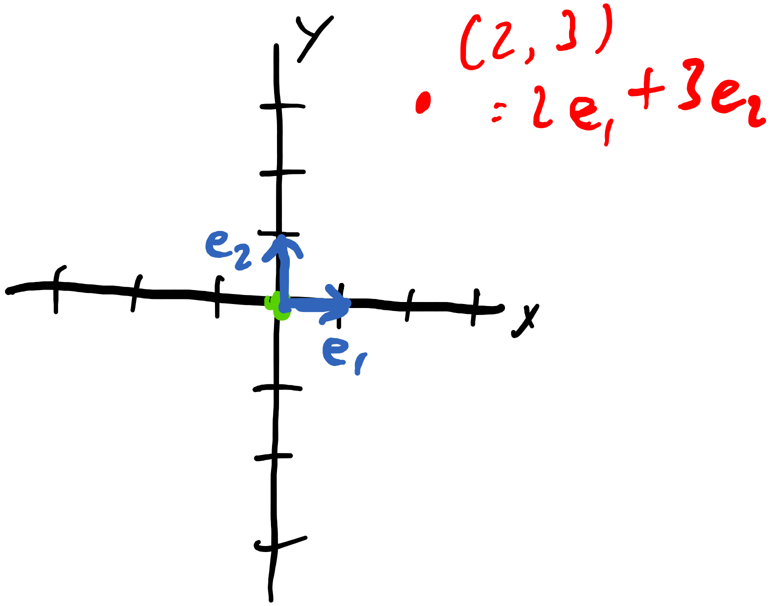
5
4
3
2
1

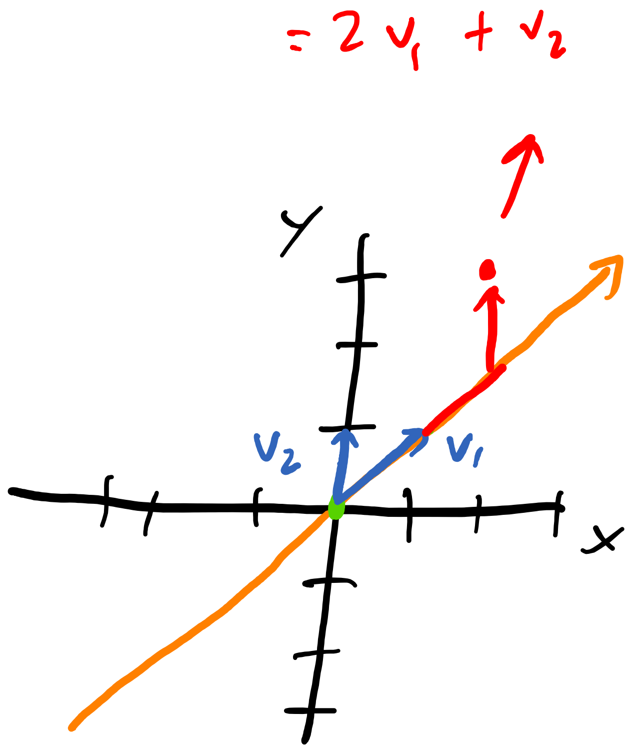
1 2 3 4 5


IC 343
(2,3)
1
2
3
4
5
Military Trail
Pan Am Drive
Parking Lot H
Parking Lot G
Centennial College Parking
Centennial Circle
Military Trail at Pan Am Drive
1065
1029
755
Instructional Centre
1063
2nd Instructional Centre UTSC
Ellesmere Road
Ellesmere & Military Trail
Morningside Avenue
Ellesmere Avenue
Conlins Road
Mirror Court
Family Mart Plus
3290
3325
3301
3348
3359
1274
1272 Military Trail
1290
1292
1265
Juniper
Larch
Koa
Maple
Rock Walk
BikeShare
Bladen Wing
UTSC Bus Loop Dropoff Point
Parking Lot A
Old Ring Road
Beechgrove Drive
Science Research Building
P105
P104
P103
P102
P101
Scholars Walk
Humanities Wing
Lot C
Residence Centre
Elm
Birch
Cedar
Dogwood
Grey Pine
Ironwood
Hickory
Joan Foley Hall
Scarborough - Rouge Park

Specifying map coordinates

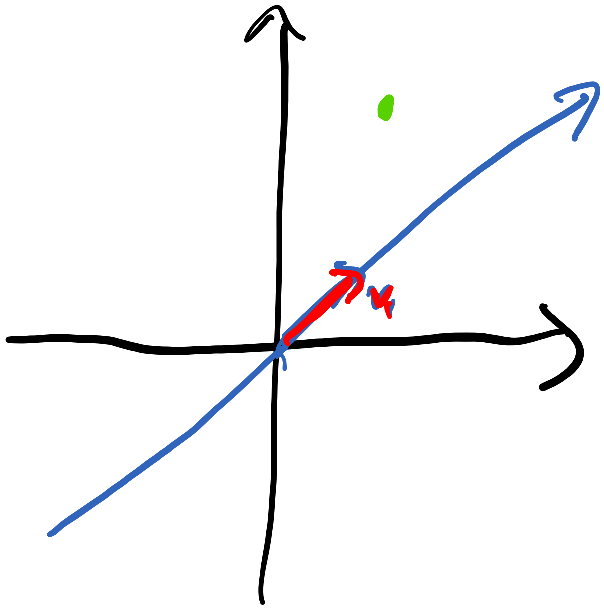
- On a 2-dimensional map, to specify a location, we need an origin point, two (independent) basic directions, and two coordinates.

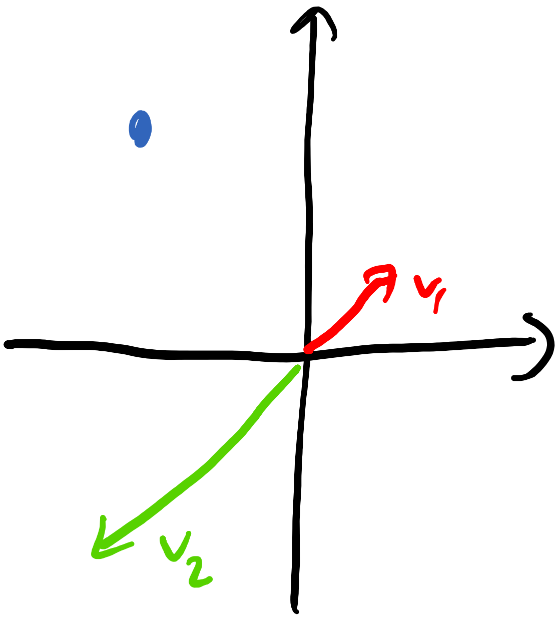




Independence of directions

- If we don't have enough basic directions (e.g. only one in 2D) or if the two basic directions chosen are not independent, then we cannot specify the location of any point (i.e. we do not have a basis)





Standard basis vectors of $\mathbb{R}^n$

- The standard basis of $\mathbb{R}^n$ is $e_1, \dots, e_n$, where e_i is the vector with all 0's except a 1 in the ith entry.
- Any vector can be written as a *linear combination* of e_i's.

R^3:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = 5e_1 + 2e_2 - e_3$$

Other sets of basis vectors of $\mathbb{R}^n$

- A set of $v_1, \dots, v_n$ is a basis of $\mathbb{R}^n$ if every vector $w \in \mathbb{R}^n$ can be written as a linear combination $w = c_1 v_1 + \cdots + c_n v_n$.
- Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$. A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

Ex. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$

Suppose $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $-1 = c = 1$ contradiction

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent

Note $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x+y}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x-y}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Test for linear independence

- The vectors $v_1, \ldots, v_n$ are linearly independent if $c_1 v_1 + \cdots c_n v_n = 0$ implies that $c_1 = \cdots = c_n = 0$.
  - Note: if you have n vectors in a space that is $< n$ dimensions, then they are not linearly independent.

Ex.

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix} \qquad c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\-2\end{bmatrix} = 0$$

$$\begin{bmatrix}1 & 2\\1 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & 0\\1 & -2 & 0\end{array}\right] \xrightarrow{R_2 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c}1 & 2 & 0\\0 & 4 & 0\end{array}\right] \xrightarrow{R_2 \leftarrow \frac{R_2}{4}} \left[\begin{array}{cc|c}1 & 2 & 0\\0 & 1 & 0\end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\end{array}\right]$$

$c_1 = 0$

$c_2 = 0$

Example

$$\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 0\\2 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{cases} c_1 + c_2 = 0 \\ 2c_1 + c_3 = 0 \end{cases}$$

$$\begin{aligned} c_2 &= -c_1 \\ c_3 &= -2c_1 \end{aligned} \Rightarrow \begin{bmatrix}c_1\\-c_1\\-2c_1\end{bmatrix} \rightarrow \begin{bmatrix}1\\-1\\-2\end{bmatrix}$$

If $c_1 = 1$, $c_2 = -1$, $c_3 = -2$, then

we get 0. So NOT lin. ind.

Try it out: are the following a basis?

- $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}$ No
- $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-5\\-4\end{bmatrix}$ Yes
- $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-5\\-4\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ No
- $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ No
- $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ Yes
- $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$ No

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Diagonal matrices scale standard basis

- Scaling operations: $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix}$

- An $n \times n$ diagonal matrix A with diagonal entries $\lambda_1, \dots, \lambda_n$ scales the *standard basis* vectors $e_1, \dots, e_n$, where e_i is a vector with 0's everywhere except a 1 in position i by $Ae_i = \lambda_i e_i$.

Matrices transform vectors

- The columns of a matrix A tell you where the matrix maps Ae_i to, where e_i are the standard basis vectors, but repeated application of A is nontrivial.

Do non-diagonal matrices scale?

- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$

Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix, and let v be a non-zero vector of length n. Then if $Av = \lambda v$ for some number λ, then v is an eigenvector of A with corresponding eigenvalue λ. Together, they are also sometimes known as an eigenpair (λ, v).
  - An eigenvector v is a vector that gets scaled by a constant multiple λ (called an eigenvalue) when multiplied by A.
  - If v is an eigenvector for the eigenvalue λ, then so is kv, for any $k \neq 0$.