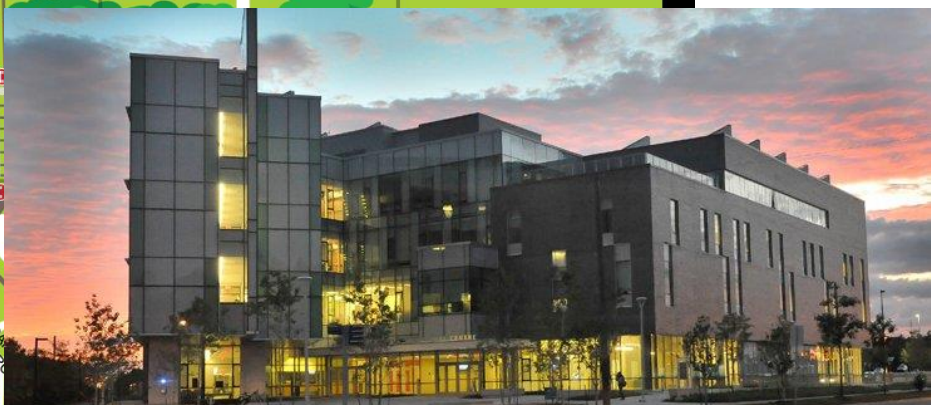


What is a basis?

Lecture 4b: 2023-01-30

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu





Specifying map coordinates

- On a 2-dimensional map, to specify a location, we need an origin point, two (independent) basic directions, and two coordinates.

Independence of directions

- If we don't have enough basic directions (e.g. only one in 2D) or if the two basic directions chosen are not independent, then we cannot specify the location of any point (i.e. we do not have a basis)

Standard basis vectors of \mathbb{R}^n

- The standard basis of \mathbb{R}^n is e_1, \dots, e_n , where e_i is the vector with all 0's except a 1 in the i th entry.
- Any vector can be written as a *linear combination* of e_i 's.

Other sets of basis vectors of \mathbb{R}^n

- A set of v_1, \dots, v_n is a basis of \mathbb{R}^n if every vector $w \in \mathbb{R}^n$ can be written as a linear combination $w = c_1 v_1 + \dots + c_n v_n$.
- Any linearly independent set of n vectors in \mathbb{R}^n is a basis of \mathbb{R}^n . A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

Test for linear independence

- The vectors v_1, \dots, v_n are linearly independent if $c_1 v_1 + \dots + c_n v_n = 0$ implies that $c_1 = \dots = c_n = 0$.
 - Note: if you have n vectors in a space that is $< n$ dimensions, then they are not linearly independent.

Example

Try it out: are the following a basis?

- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

A: Yes

B: No

C: Maybe

D: ???

E: None of the above

Diagonal matrices scale standard basis

• Scaling operations:
$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}^A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix}$$

- An $n \times n$ diagonal matrix A with diagonal entries $\lambda_1, \dots, \lambda_n$ scales the *standard basis* vectors e_1, \dots, e_n , where e_i is a vector with 0's everywhere except a 1 in position i by $Ae_i = \lambda_i e_i$.

Matrices transform vectors

- The columns of a matrix A tell you where the matrix maps Ae_i to, where e_i are the standard basis vectors, but repeated application of A is nontrivial.

Do non-diagonal matrices scale?

- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix}$

Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix, and let v be a non-zero vector of length n . Then if $Av = \lambda v$ for some number λ , then v is an eigenvector of A with corresponding eigenvalue λ . Together, they are also sometimes known as an eigenpair (λ, v) .
 - An eigenvector v is a vector that gets scaled by a constant multiple λ (called an eigenvalue) when multiplied by A .
 - If v is an eigenvector for the eigenvalue λ , then so is kv , for any $k \neq 0$.