Eigenvalues, eigenvectors, and eigenbases
Lecture 4c: 2023-02-02

MAT A35 – Winter 2023 – UTSC

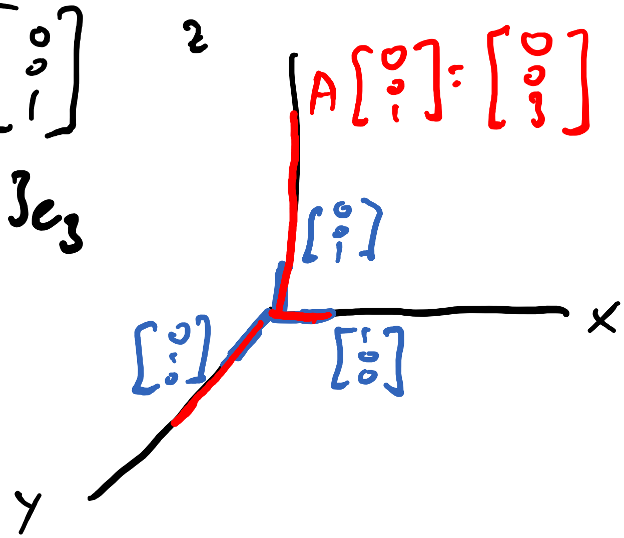
Prof. Yun William Yu

Diagonal matrices scale standard basis

- Scaling operations: $\overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}}^{A}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}x\\ 2y\\ 3z\end{bmatrix}$

Basis vectors: $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$Ae_1 = e_1 \qquad Ae_2 = \begin{bmatrix}0\\2\\0\end{bmatrix} = 2e_2 \qquad Ae_3 = 3e_3$



- An $n \times n$ diagonal matrix A with diagonal entries $\lambda_1, \dots, \lambda_n$ scales the *standard basis* vectors $e_1, \dots, e_n$, where e_i is a vector with 0's everywhere except a 1 in position i by $Ae_i = \lambda_i e_i$.

Matrices transform vectors

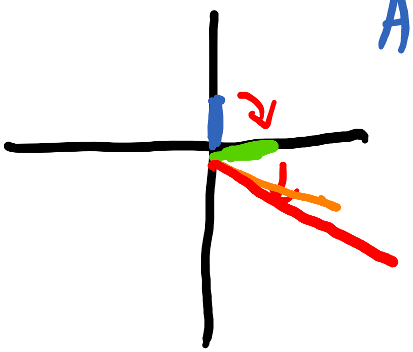
- The columns of a matrix A tell you where the matrix maps Ae_i to, where e_i are the standard basis vectors, but repeated application of A is nontrivial.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$Ae_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

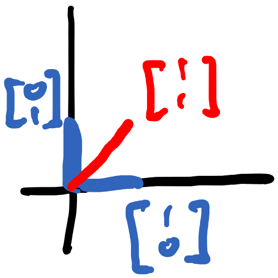
$$Ae_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

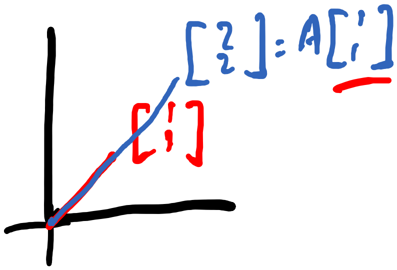
$$A[Ae] = A^2 e_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

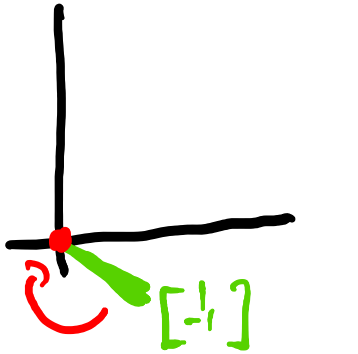


Do non-diagonal matrices scale?

- $\overbrace{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}}^{A}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x+y\\ x+y\end{bmatrix}$







$$\underbrace{A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = A\begin{bmatrix}1\\0\end{bmatrix}}_{\text{does not scale } \begin{bmatrix}0\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\0\end{bmatrix}}$$

$$\underbrace{A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\cdot\begin{bmatrix}1\\1\end{bmatrix}}_{\text{scales } \begin{bmatrix}1\\1\end{bmatrix} \text{ by a factor of } 2}$$

$$\underbrace{A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}}_{\text{Scales } \begin{bmatrix}1\\-1\end{bmatrix} \text{ by } 0}$$

Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix, and let v be a non-zero vector of length n. Then if $Av = \lambda v$ for some number λ, then v is an eigenvector of A with corresponding eigenvalue λ. Together, they are also sometimes known as an eigenpair (λ, v).
  - An eigenvector v is a vector that gets scaled by a constant multiple λ (called an eigenvalue) when multiplied by A.
  - If v is an eigenvector for the eigenvalue λ, then so is kv, for any $k \neq 0$.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenpairs $\boxed{\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(0, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}$

Also works $\left(2, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right), \left(0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

Try it out

- Let $A = \begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix}$.
- Which of the following are eigenvectors of A?

A: $[2 \quad -6 \quad 5]^T$
B: $[6 \quad 1 \quad 3]^T$
C: $[12 \quad 2 \quad 6]^T$
D: All of the above
E: None of the above

$$\begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -18 - 36 + 100 \\ 4 - 12 - 20 \\ -12 - 18 + 65 \end{bmatrix} = \begin{bmatrix} 46 \\ -28 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix} \begin{bmatrix} 12 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -108 + 12 + 120 \\ 24 + 4 - 24 \\ -72 + 6 + 78 \end{bmatrix} = \begin{bmatrix} 24 \\ 4 \\ 12 \end{bmatrix}$$

$2\times$

Finding eigenvalues of a matrix

- Let A be a $n \times n$ matrix. If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$.

proof.

$Av = \lambda v$ for some nonzero v.

$Av - \lambda v = 0$

$Av - \lambda(Iv) = 0$ (because $Iv = v$)

$(A - \lambda I)v = 0$ (distributive prop)

$\Rightarrow A - \lambda I$ is a singular matrix (since $v \neq 0$)

$\Rightarrow \det(A - \lambda I) = 0$

Example

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 - 1$$

$$= 1 - 2\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\underbrace{\lambda = 0, \quad \lambda = 2}_{\text{eigenvalues}}$$

Try it out

$\det(A - \lambda I) = 0$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1$

Find the eigenvalues:
A: 1
B: 2
C: 3
D: All of the above
E: None of the above

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 6 = 0$

$\lambda^2 - 5\lambda + 4 - 6 = 0$

$\lambda^2 - 5\lambda - 2 = 0$

$\lambda = \frac{5 \pm \sqrt{25+8}}{2}$

$= \frac{1}{2}(5 \pm \sqrt{33})$

$\lambda_1 = \frac{5}{2} + \frac{\sqrt{33}}{2}$

$\lambda_2 = \frac{5}{2} - \frac{\sqrt{33}}{2}$

Find the eigenvalues:
A: 1
B: 2
C: 3
D: All of the above
E: None of the above

Try it out

- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Find the eigenvalues:
A: 1
B: 2
C: 3
D: All of the above
E: None of the above

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 2, 3$$

- Triangular matrices have their eigenvalues on the diagonal.

Finding eigenvectors of a matrix

- $Av = \lambda v$, or alternately, $\underline{(A - \lambda I)v = 0}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$(\lambda + 1)(\lambda - 1) = 0$

$\Rightarrow \lambda = 1, -1 \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1$

$\lambda_1 = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x - y = 0 \Rightarrow x = y$

$v_1 = \begin{bmatrix} x \\ x \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x + y = 0 \Rightarrow y = -x$

$v_2 = \begin{bmatrix} x \\ -x \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Example

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$\boxed{\lambda_1 = 1}$

$\lambda_2 = 2$

$\lambda_3 = 3$

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left.\begin{aligned} x + 4y + 6z &= x \\ 2y + 5z &= y \\ 3z &= z \end{aligned}\right\}$$

$$\left\{\begin{aligned} 4y + 6z &= 0 \\ y + 5z &= 0 \\ 2z &= 0 \end{aligned}\right.$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_3 = \frac{R_3}{2}$

$$\left[\begin{array}{ccc|c} 0 & 4 & 6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 = R_1 - 6R_3$

$R_2 = R_2 - 5R_3$

$$\left[\begin{array}{ccc|c} 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 = R_1 - 4R_2$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} y &= 0 \\ z &= 0 \end{aligned}$$

$$V_1 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$x = 1$

Example (continued) $\lambda_2 = 2$

$$\begin{bmatrix}1&4&6\\0&2&5\\0&0&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2x\\2y\\2z\end{bmatrix}$$

$$\begin{cases} x+4y+6z=2x\\ 2y+5z=2y\\ 3z=2z\end{cases}$$

$$z=0$$

$$\begin{cases} x+4y=2x\\ 2y=2y\end{cases}$$

$$x=4y$$
$$y=y$$

$$v_2=\begin{bmatrix}4y\\y\\0\end{bmatrix}\xrightarrow[\text{set } y=1]{}\begin{bmatrix}4\\1\\0\end{bmatrix}$$

Try it out

- $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$. What is the eigenvector corresponding to the eigenvalue $\lambda = 3$?

$$x + 4y + 6z = 3x$$
$$2y + 5z = 3y$$
$$\underbrace{3z = 3z}_{z = z}$$

$$y = 5z$$

$$x + 20z + 6z = 3x$$
$$2x = 26z$$
$$x = 13z$$

$$v_3 = \begin{bmatrix} 13z \\ 5z \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} 13 \\ 5 \\ 1 \end{bmatrix}$$

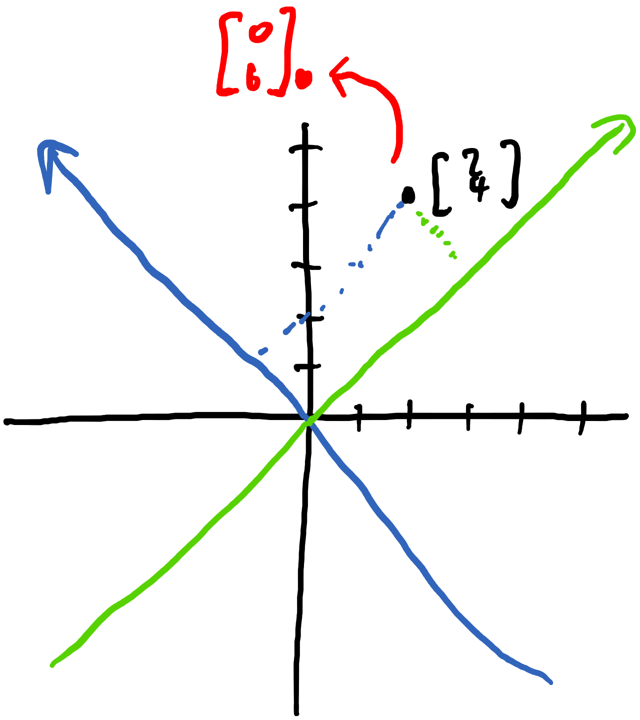
Find the corresponding eigenvector:

A: $[0, 0, 1]^T$
B: $[0, 5, 1]^T$
C: $[13,5,1]^T$
D: All of the above
E: None of the above

Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the "action" of A by what it does to the eigenvectors.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has eigenpairs} \quad \left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \quad \left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$



Note $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot A \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the "action" of A by what it does to the eigenvectors.

Eigenbasis of a square matrix

- If an $n \times n$ matrix A has n linearly independent eigenvectors, those eigenvectors form an eigenbasis.

$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ has eigenbasis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so not an eigenbasis

- Note that eigenvectors corresponding to different eigenvalues are necessarily linearly independent.

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

Suppose $v_1 = c_1 v_2$

then $Av_1 = c_1 A v_2$

$\Rightarrow \lambda_1 v_1 = c_1 \lambda_2 v_2$

$\Rightarrow \lambda_1 c_1 v_2 = \lambda_2 c_1 v_2$

$\Rightarrow \lambda_1 = \lambda_2$

- Also, can find all linearly independent eigenvectors corresponding to an eigenvalue by setting each of the free variables after Gaussian elimination.

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \Rightarrow$

$2x = 2x$

$2y = 2y$

$\underbrace{z = 2z}_{\to z=0}$

$v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$,

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Try it out: do the following have an eigenbasis?

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Yes.

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$\Rightarrow 2x = 2x$
$2y = 2y$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Yes 2 different eigenvalues

- $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ No

$\lambda = 2$

$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$2x = 2x$
$x + 2y = 2y$
$x = 0$

$\begin{bmatrix} 0 \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Yes

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = 0$

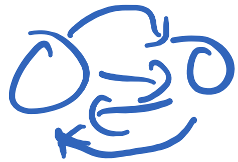
$4 - 5\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 5\lambda = 0$

$\lambda(\lambda - 5) = 0$

$\lambda = 0, 5$

Population Growth Rates



- Suppose that the Leslie matrix L for a population has eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. If the initial population vector is $p = a_1 v_1 + \cdots + a_n v_n$, then the population after t time periods is

$$a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$$

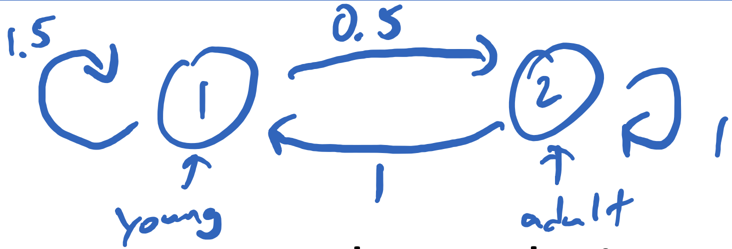
proof. The pop vector after t time is $L^t p$

$$= L^t [a_1 v_1 + \cdots + a_n v_n]$$

$$= a_1 L^t v_1 + \cdots + a_n L^t v_n$$

$$= a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$$

Example



- Consider an age-structured population model for birds where you have divided the group into young and old. Each old has only 1 hatchling each year, but survives with probability 1. Each young has 1.5 new hatchlings each year, but survives with only probability 0.5 to become old next year. If $p_0 = [6, 0]^T$, what is the population after 10 years?

Leslie matrix $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$

$p_0 = \begin{bmatrix} \text{young} \\ \text{old} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ population vector

Need to find $L^{10} p_0 = p_{10}$

Eigendecomposition of $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$

Eigenvalues

$$\begin{vmatrix} 1.5-\lambda & 1 \\ 0.5 & 1-\lambda \end{vmatrix} = 1.5 - 2.5\lambda + \lambda^2 - 0.5 = 0$$

$$\lambda^2 - 2.5\lambda + 1 = 0$$

$$(\lambda - 2)(\lambda - 0.5) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 0.5$$

Eigenvectors: $\underline{\lambda_1 = 2}$

$$\begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ \underline{2y} \end{bmatrix}$$

$$\begin{cases} 1.5x + y = 2x \\ 0.5x + y = 2y \end{cases} \qquad 0.5x = y$$

$$v_1 = \begin{bmatrix} x \\ 0.5x \end{bmatrix} \to \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = 0.5}$

$$\begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ \underline{0.5y} \end{bmatrix}$$

$$\begin{cases} 1.5x + y = 0.5x \\ 0.5x + y = 0.5y \end{cases}$$

$$x + y = 0 \qquad x = -y$$

$$v_2 = \begin{bmatrix} -y \\ y \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Rewrite $\begin{bmatrix} 6 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{array}{l} 2c_1 - c_2 = 6 \\ c_1 + c_2 = 0 \end{array} \right.$$

$$\left[\begin{array}{cc|c} 2 & -1 & 6 \\ 1 & 1 & 0 \end{array} \right]$$

one method

$$3c_1 = 6$$

$$\boxed{\begin{array}{l} c_1 = 2 \\ c_2 = -2 \end{array}}$$

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solve $\mathbf{p}_{10} = L^{10}p_0$ using eigenvectors

$$p_{10} = L^{10}\begin{bmatrix}6\\0\end{bmatrix} = L^{10}\left[2\cdot \overset{\lambda=2}{\begin{bmatrix}2\\1\end{bmatrix}} - 2\overset{\lambda=0.5}{\begin{bmatrix}-1\\1\end{bmatrix}}\right]$$

$$= 2\cdot\left(L^{10}\begin{bmatrix}2\\1\end{bmatrix}\right) - 2\left(L^{10}\begin{bmatrix}-1\\1\end{bmatrix}\right)$$

$$= 2\cdot\left(2^{10}\begin{bmatrix}2\\1\end{bmatrix}\right) - 2\left(\underbrace{0.5^{10}}_{\frac{1}{2^{10}}}\begin{bmatrix}-1\\1\end{bmatrix}\right)$$

$$= 2^{11}\begin{bmatrix}2\\1\end{bmatrix} - \frac{1}{2^9}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}4096\\2048\end{bmatrix} - \frac{1}{2^9}\begin{bmatrix}-1\\1\end{bmatrix} \approx \begin{bmatrix}4096\\2048\end{bmatrix}$$

Example

- What if the initial population were $p_0 = [3, 0]^T$?

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow a_1 = 1 \qquad a_2 = -1$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$p_{10} = 2^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 0.5^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2048 \\ 1024 \end{bmatrix} - \begin{bmatrix} -\frac{1}{1024} \\ \frac{1}{1024} \end{bmatrix}$$

$$\approx \begin{bmatrix} 2048 \\ 1024 \end{bmatrix}$$

Try it out

$$\begin{cases} 0 = 2a_1 - a_2 \\ 6 = a_1 + a_2 \end{cases}$$

- What if the initial population size was $p_0 = [0, 6]^T$? Which of the following answers is the closest to the population vector after 10 years?
- Recall $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 0.5$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

A: $[1000, 500]^T$
B: $[2000, 1000]^T$
C: $[4000, 2000]^T$
D: $[8000, 4000]^T$
E: $[16000, 8000]^T$

$$P_{10} = 2 \cdot 2^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4 \cdot 0.5^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 4096 \\ 2048 \end{bmatrix}$$

- Note that because exponentials grow super-fast, the long-term growth rate is dominated by the largest eigenvalue.

Try it out

- Consider a population with three life stages, newborn, juvenile, and adult, with the Leslie matrix

$$L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

At long time scales, what is the ratio of newborns to adults?

$$\begin{vmatrix} -\lambda & 6 & 8 \\ 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \end{vmatrix} = -\lambda^3 + 2 + 3\lambda = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-2) = 0$$

$$\lambda = \underline{2}, -1$$

$$\begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ \underline{2z} \end{bmatrix}$$

$$\begin{cases} 6y + 8z = 2x \\ 0.5x = 2y \\ 0.5y = 2z \end{cases}$$

$$y = 4z$$

$$x = 4y = 16z$$

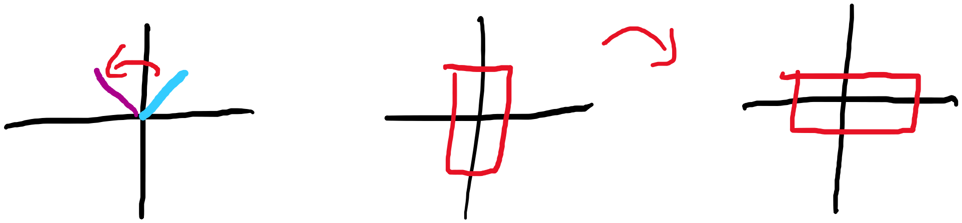
$$v = \begin{bmatrix} 16z \\ 4z \\ z \end{bmatrix} \to \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

A: 2 to 1
B: 4 to 1
C: 8 to 1
D: 16 to 1
E: None

Advanced topic: complex eigenpairs

- **Note that this will NOT be on Quiz 2.**
- We've talked a lot about scaling by a constant multiple. But what happens if the numbers aren't real?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1, \quad \lambda = \pm i$$

- It turns out that imaginary eigenvalues correspond to rotations.
- Complex eigenvalues can be a combination of scaling and rotation.