Eigenvalues, eigenvectors, and eigenbases Lecture 4c: 2023-02-02

MAT A35 – Winter 2023 – UTSC

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Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix, and let v be a non-zero vector of length n. Then if $Av = \lambda v$ for some number λ, then v is an eigenvector of A with corresponding eigenvalue λ. Together, they are also sometimes known as an eigenpair (λ, v).
  - An eigenvector v is a vector that gets scaled by a constant multiple λ (called an eigenvalue) when multiplied by A.
  - If v is an eigenvector for the eigenvalue λ, then so is kv, for any $k \neq 0$.

Try it out

- Let $A = \begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix}$.
- Which of the following are eigenvectors of A?

A: $[2 \quad -6 \quad 5]^T$
B: $[6 \quad 1 \quad 3]^T$
C: $[12 \quad 2 \quad 6]^T$
D: All of the above
E: None of the above

Finding eigenvalues of a matrix

- Let A be a $n \times n$ matrix. If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$.

Example

Try it out

- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Find the eigenvalues:
A: 1
B: 2
C: 3
D: All of the above
E: None of the above

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Find the eigenvalues:
A: 1
B: 2
C: 3
D: All of the above
E: None of the above

Try it out

- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Find the eigenvalues:
A: 1
B: 2
C: 3
D: All of the above
E: None of the above

- Triangular matrices have their eigenvalues on the diagonal.

Finding eigenvectors of a matrix

- $Av = \lambda v$, or alternately, $(A - \lambda I)v = 0$

Example

Example (continued)

Try it out

- $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$. What is the eigenvector corresponding to the eigenvalue $\lambda = 3$?

Find the corresponding eigenvector:

A: $[0, 0, 1]^T$

B: $[0, 5, 1]^T$

C: $[13,5,1]^T$

D: All of the above

E: None of the above

Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the “action” of A by what it does to the eigenvectors.

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Eigenbasis of a square matrix

- If an $n \times n$ matrix A has n linearly independent eigenvectors, those eigenvectors form an eigenbasis.

- Note that eigenvectors corresponding to different eigenvalues are necessarily linearly independent.

- Also, can find all linearly independent eigenvectors corresponding to an eigenvalue by setting each of the free variables after Gaussian elimination.

Try it out: do the following have an eigenbasis?

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Population Growth Rates

- Suppose that the Leslie matrix G for a population has eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. If the initial population vector is $p = a_1 v_1 + \cdots + a_n v_n$, then the population after t time periods is
$$a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$$

Example

- Consider an age-structured population model for birds where you have divided the group into young and old. Each old has only 1 hatchling each year, but survives with probability 1. Each young has 1.5 new hatchlings each year, but survives with only probability 0.5 to become old next year. If $p_0 = [6, 0]^T$, what is the population after 10 years?

Eigendecomposition of $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$

Rewrite $\begin{bmatrix} 6 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solve $\mathrm{p}_{10} = L^{10} p_0$ using eigenvectors

Example

- What if the initial population were $p_0 = [3, 0]^T$?

Try it out

- What if the initial population size was $p_0 = [0, 6]^T$? Which of the following answers is the closest to the population vector after 10 years?
- Recall $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 0.5$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A: $[1000, 500]^T$
B: $[2000, 1000]^T$
C: $[4000, 2000]^T$
D: $[8000, 4000]^T$
E: $[16000, 8000]^T$

- Note that because exponentials grow super-fast, the long-term growth rate is dominated by the largest eigenvalue.

Try it out

- Consider a population with three life stages, newborn, juvenile, and adult, with the Leslie matrix

$$L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

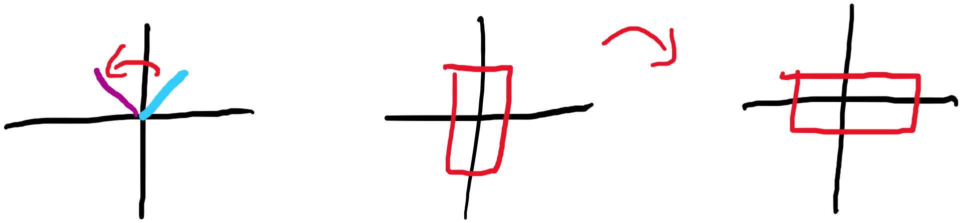
At long time scales, what is the ratio of newborns to adults?

A: 2 to 1
B: 4 to 1
C: 8 to 1
D: 16 to 1
E: None

Advanced topic: complex eigenpairs

- **Note that this will NOT be on Quiz 2.**
- We've talked a lot about scaling by a constant multiple. But what happens if the numbers aren't real?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1, \quad \lambda = \pm i$$

- It turns out that imaginary eigenvalues correspond to rotations.
- Complex eigenvalues can be a combination of scaling and rotation.