

Functions of several variables

Lecture 5a: 2023-02-06

MAT A35 – Winter 2023 – UTSC

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What is a function?

- A function $f: V \rightarrow W$ takes an input in V and gives a (single) output in W .
- Easiest example is when $V = W = \mathbb{R}$, i.e. both are real numbers.

- Another classic example is when $V = W = \mathbb{C}$, both complex numbers.

- We also have less “mathematical” examples. Let V be the set of days, and let W be the set of emotions, and let $f: V \rightarrow W$ be your dominant emotion on that day.

Try it out: is this a function?

- $f: [\text{set of persons}] \rightarrow [\text{set of colors}]$, where given a person, f tells you what their favorite color is (assuming each person has exactly 1 favorite color)
- $g: [\text{set of persons}] \rightarrow [\text{set of colors}]$, where given a person, g tells you all the colors they like (can be multiple).
- $h: [\text{set of persons}] \rightarrow [\text{set of all sets of colors}]$, where given a person, h tells you all the colors they like (can be multiple).
- $r: [\text{photos on Instagram}] \rightarrow \{0,1\}$, where r returns 1 if the photo has a cat, and 0 if the photo does not have a cat.



) = 1,



$r($

) = 0

- $f: \mathbb{R} \rightarrow \mathbb{R}$, where $f(x) = \pm\sqrt{x}$
- $g: \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = \sqrt{x}$
- $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $h(v) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} v$

A: Yes

B: No

C: Maybe

D: ???

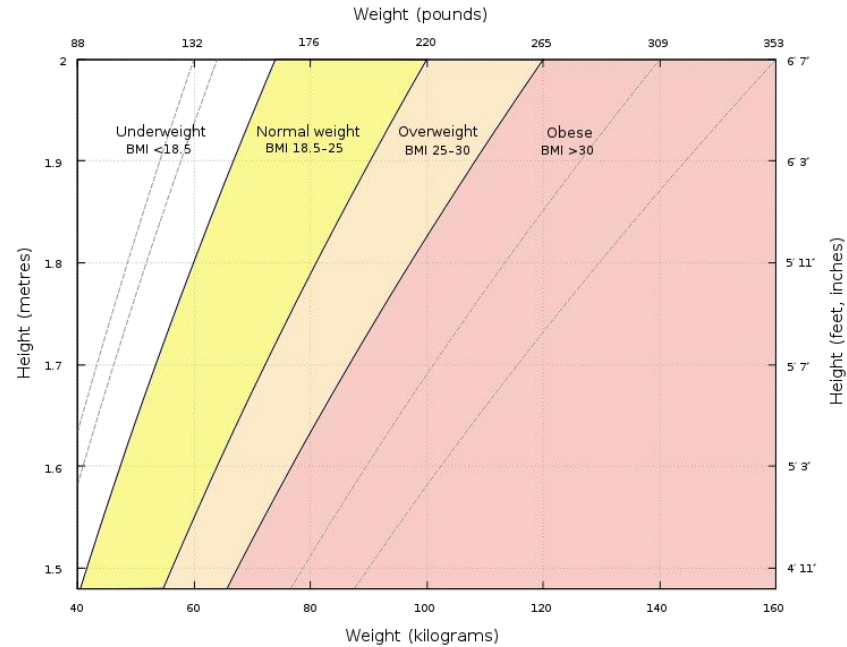
E: None of the above

Functions of two variables

- Given $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, we know that given any 2D vector $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathbb{R}$ are real numbers and the output $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is another real number.
 - Often, we will write $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ as $f(x, y)$ for convenience, so f can be thought of as a function of two real variables.

Application – body mass index (BMI)

- The Body-Mass Index (BMI) was developed by Adolphe Quetelet to (approximately) quantify obesity.
- $B(m, h) = \frac{m}{h^2}$, where m is mass in kilograms and h is height in meters.



https://commons.wikimedia.org/wiki/File:BMI_chart.svg

When was BMI invented?

- A: Before 1700 A.D.
- B: 1700-1800 A.D.
- C: 1800-1900 A.D.
- D: 1900-2000 A.D.
- E: After 2000 A.D.

Functions of several variables

- $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function from n real numbers that outputs 1 real number.

Geometric interpretation of functions

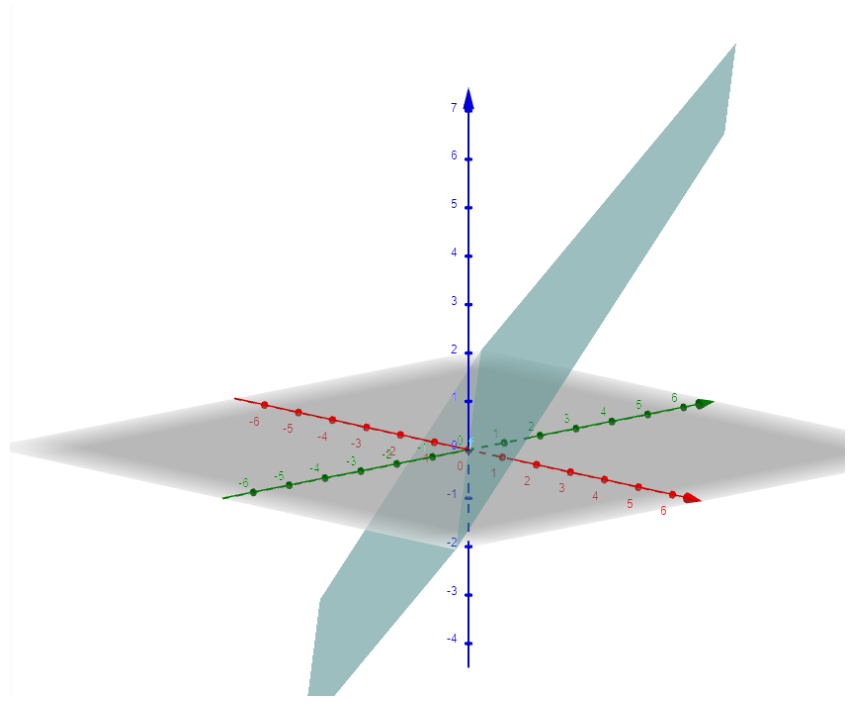
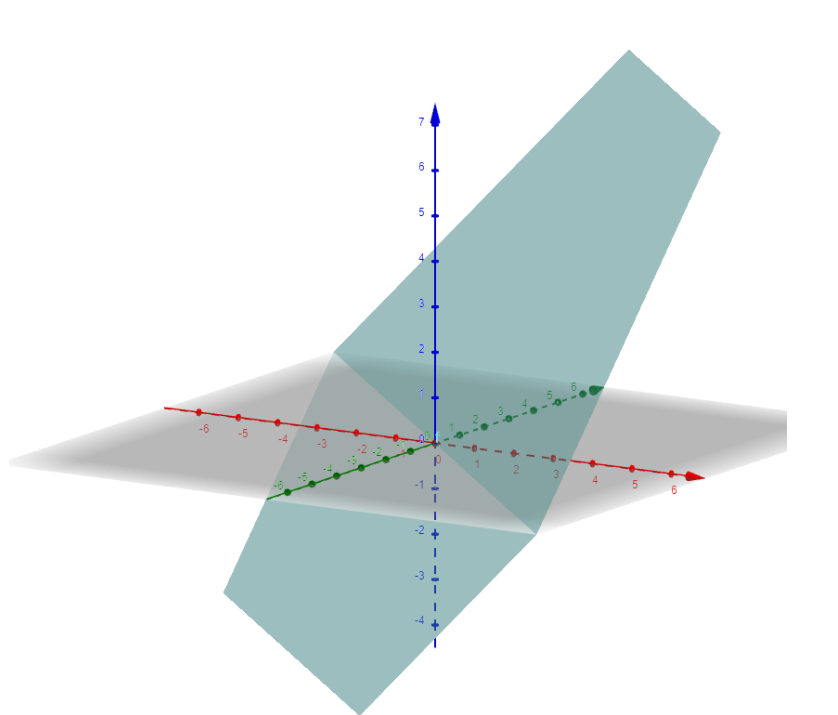
- $f: \mathbb{R} \rightarrow \mathbb{R}$ “maps” a real number to a real number.
- We can think of the function as pairs of numbers in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, which is what we do when we draw the graph of a function.

Geom. interpret. of 2-variable function

- $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ “maps” a pair of numbers to a single number.
- We can think of the function as a list of ordered pairs $((x, y), z) \in \mathbb{R}^2 \times \mathbb{R}$, where the first part of the ordered pair is a pair $(x, y) \in \mathbb{R}^2$ itself. We can then “graph” the function by drawing a surface in \mathbb{R}^3 .

3D plotting

- <https://www.geogebra.org/3d?lang=en>
 - $z = 0$
 - $f(x, y) = x + y$



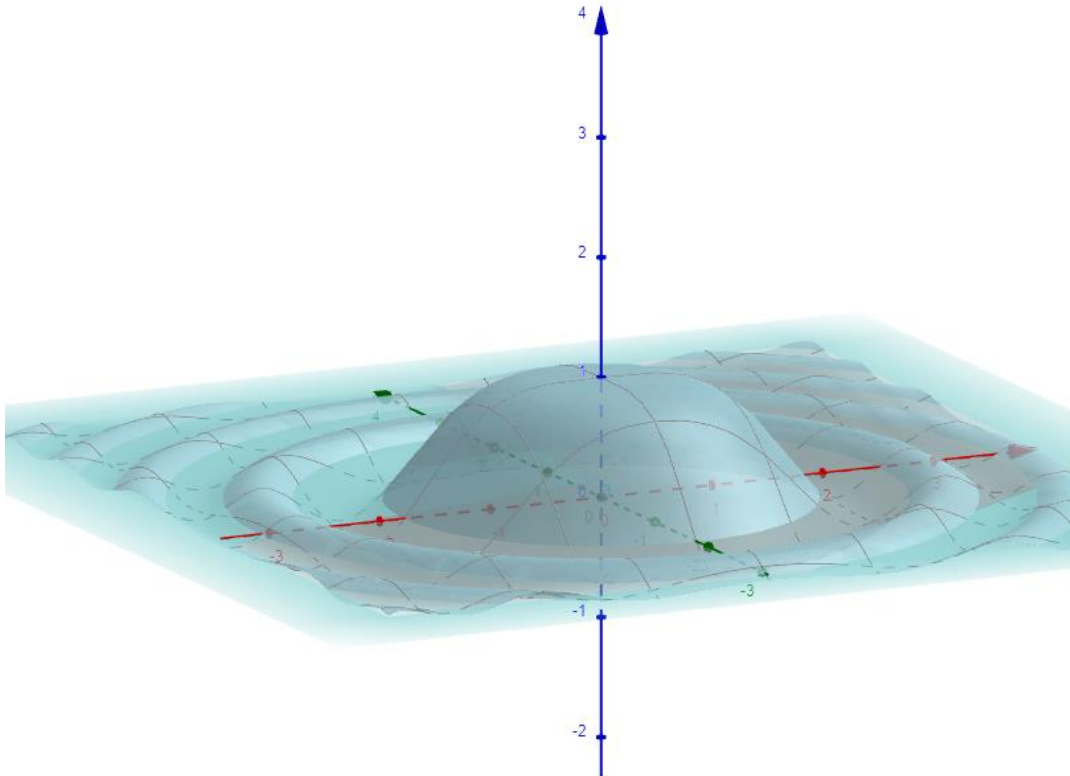
More 3D plotting examples

- <https://www.geogebra.org/3d?lang=en>

- $z = 0$

- $f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$

$$f(x, y) = \sin(x^2 + y^2) / (x^2 + y^2)$$



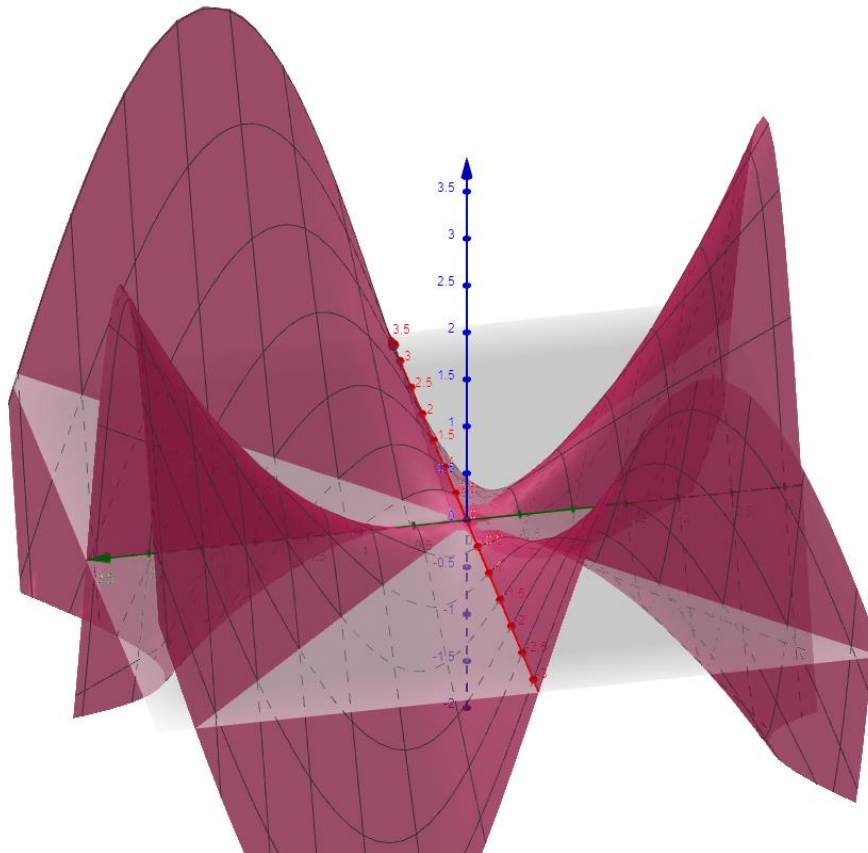
More 3D plotting examples

- <https://www.geogebra.org/3d?lang=en>

- $z = 0$

- $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$

$$f(x, y) = x * y (x^2 - y^2) / (x^2 + y^2)$$



Try it out yourself:

- <https://www.geogebra.org/3d?lang=en>

- $f(x, y) = x^2 + y^2$
 $f(x, y) = x^2 + y^2$

- $f(x, y) = 5$
 $f(x, y) = 5$

- $f(x, y) = x^2 - y^2$
 $f(x, y) = x^2 - y^2$

- $f(x, y) = x^2 + y^2 + \frac{1}{x^2 + y^2}$
 $f(x, y) = x^2 + y^2 + 1 / (x^2 + y^2)$

- $f(x, y) = \sin x + \cos y$
 $f(x, y) = \sin(x) + \cos(y)$