Multivariable integration Lecture 5c: 2023-02-~~06~~9

MAT A35 – Winter 2023 – UTSC

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Double definite integrals

$$\int_{\underbrace{a}_{\text{limits of integration}}}^{b} f(x)\underbrace{dx}_{\text{variable we integrate over}} = \int_{x=a}^{x=b} f(x)\,dx = ANS$$

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_{y=a}^{y=b}\int_{x=c}^{x=d} f(x,y)\,dx\,dy$$

int over x

int over y

Definite integration removes the variable

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]\Bigg|_{x=1}^{x=2} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_1^a x^2\,dx = \left[\frac{1}{3}x^3\right]\Bigg|_{x=1}^{x=a} = \frac{a^3}{3} - \frac{1}{3}$$

$$\int_1^y x^2\,dx = \left[\frac{1}{3}x^3\right]\Bigg|_{x=1}^{x=y} = \frac{y^3}{3} - \frac{1}{3}$$

Multiple definite integrals example

$$\int_{y=a}^{y=b}\left(\int_{x=c}^{x=d} f(x,y)\,dx\right)dy$$

- First integrate the inside integral, assuming all other variables are constant.
  - $\int_{x=c}^{x=d} f(x,y)\,dx = g(y)$ (because we got rid of the "x")
- Then integrate the outside variable, to get an answer
  - $\int_{y=a}^{y=b} g(y)\,dy =$ answer

Ex: $\int_0^2 \underbrace{\int_{-1}^2 10xy^2\,dx}\ dy$

- $\displaystyle\int_{-1}^{2} 10\,xy^2\,dx = \left[5x^2y^2\right]\Big|_{x=-1}^{x=2} = 20y^2 - 5y^2$
  $= 15y^2$

  treat y as constant

- $\displaystyle\int_0^2 15y^2\,dy = \left[5y^3\right]\Big|_{y=0}^{y=2} = \underline{\underline{40}}$

Try it out

- $\int_0^2 \int_{-1}^1 2y dx\, dy$
  - $\int_{-1}^{1} 2y\, dx = [2xy]\Big|_{x=-1}^{x=1} = 2y - (-2y) = 4y$
  - $\int_0^2 4y\, dy = [2y^2]\Big|_{y=0}^{y=2} = 8$

A: 2

B: 4

C: 8

D: 16

E: None

Switching order of integrals?

- Last slide: $\int_0^2 \int_{-1}^1 2y dx\, dy$ $= 8$
- What about: $\int_{-1}^1 \int_0^2 2y dy\, dx$

$\int_0^2 2y\, dy = y^2 \Big|_0^2 = 4$

$\int_{-1}^1 4\, dx = 4x \Big|_{-1}^1 = 4 - (-4) = 8$

- Often, we can switch the order of integration and get the same answer, but this is not always true.

Variables in the limits of integration

- We can use the outside integral variable in the limits of the inside variable (but not the other way around).

Ex.

$$\int_0^1 \int_x^{x^2} xy^2\,dy\,dx$$

- $\displaystyle\int_{y=x}^{y=x^2} xy^2\,dy = \left[\frac{1}{3}xy^3\right]\Big|_{y=x}^{y=x^2} = \frac{x^7}{3} - \frac{x^4}{3}$

- $\displaystyle\int_0^1 \left[\frac{x^7}{3} - \frac{x^4}{3}\right]dx = \frac{1}{3}\int_0^1 (x^7 - x^4)\,dx = \frac{1}{3}\left[\frac{1}{8}x^8 - \frac{1}{5}x^5\right]_{x=0}^{x=1}$

$$= \frac{1}{3}\cdot\left[\frac{1}{8} - \frac{1}{5}\right] = \frac{1}{3}\cdot\left[\frac{5}{40} - \frac{8}{40}\right] = \frac{1}{3}\cdot\frac{-3}{40} = -\frac{1}{40}$$

Try it out

- $\int_0^2 \int_1^{y^2} 2x dx\, dy$

- $\int_1^{y^2} 2x\, dx = x^2 \Big|_{x=1}^{x=y^2} = y^4 - 1$

- $\int_0^2 (y^4 - 1)\, dy = \left[\frac{1}{5} y^5 - y\right] \Big|_0^2 = \left[\frac{32}{5} - 2\right] - \left[0 - 0\right]$

$= \frac{32}{5} - 2 = \frac{22}{5}$

A: $\frac{11}{5}$

B: $\frac{22}{5}$

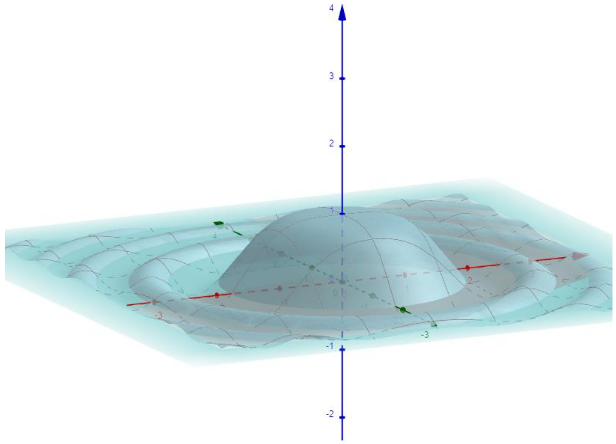
C: $\frac{33}{5}$

D: $\frac{44}{5}$

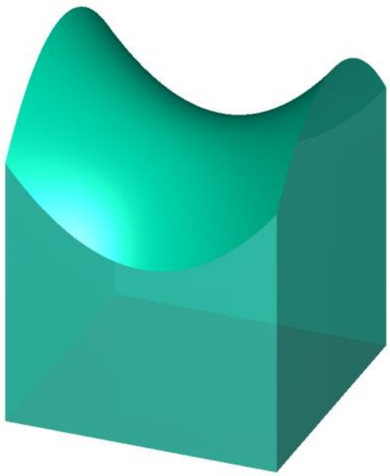
E: None

Geometric interpretation

- Multivariable functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ can be thought of as surfaces.



- Double integrals correspond to the *volume* under the surface for a particular region



https://en.wikipedia.org/wiki/Multiple_integral