Multilinear and Nonlinear Regression Lecture 6c: 2023-02-16

MAT A35 – Winter 2023 – UTSC

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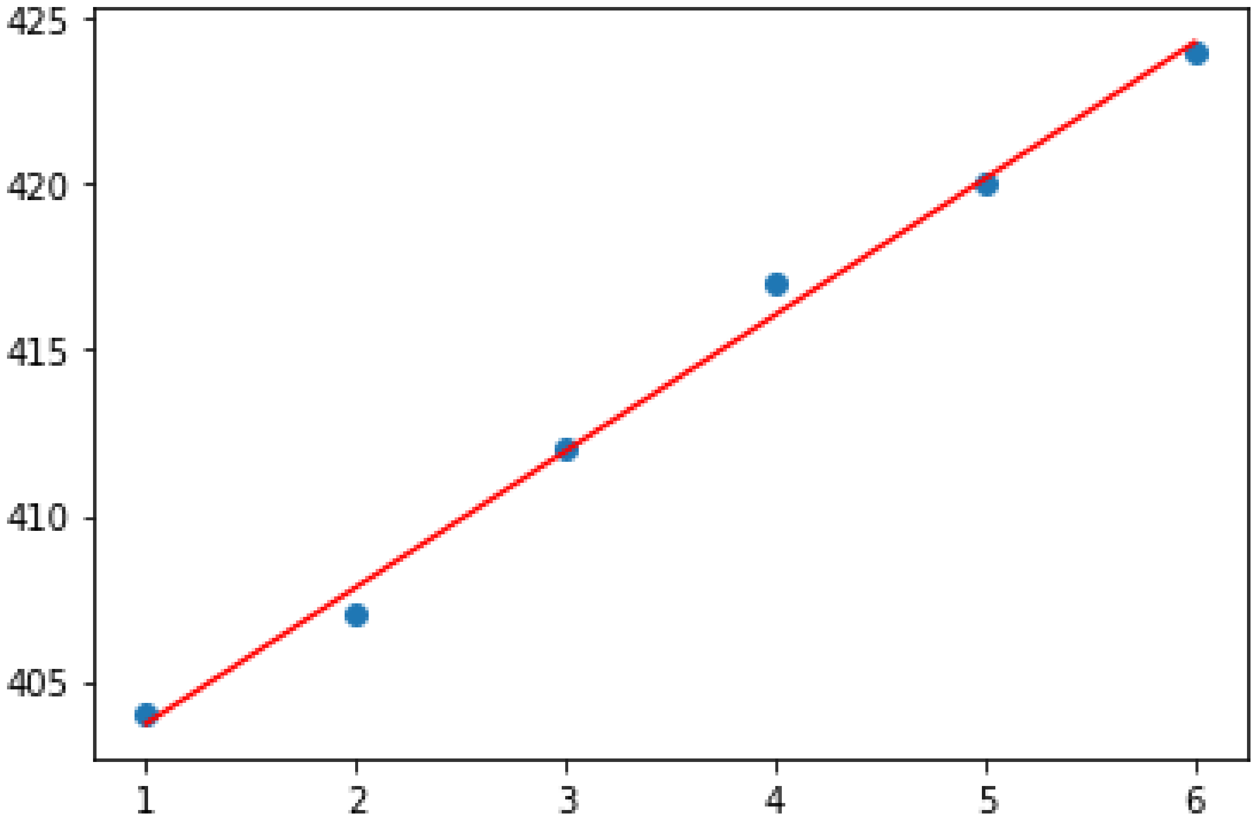
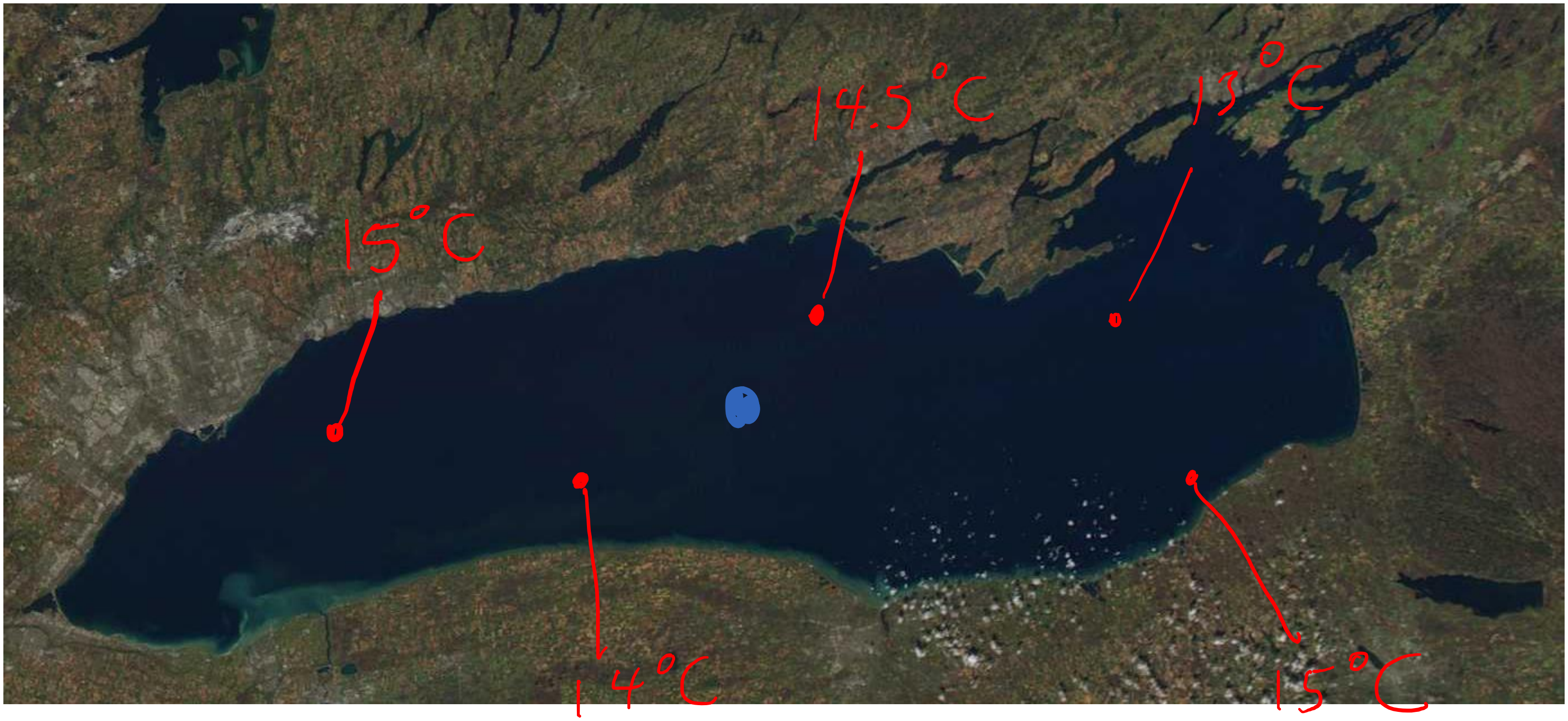
Single variable linear regression

- Given samples of the dependent variable $y_1, \ldots, y_n$ at values of the independent variable $x_1, \ldots, x_n$, we want to find the linear model $f(x) = mx + b$ such that $y_i \approx f(x_i)$, the "best-fit" line.

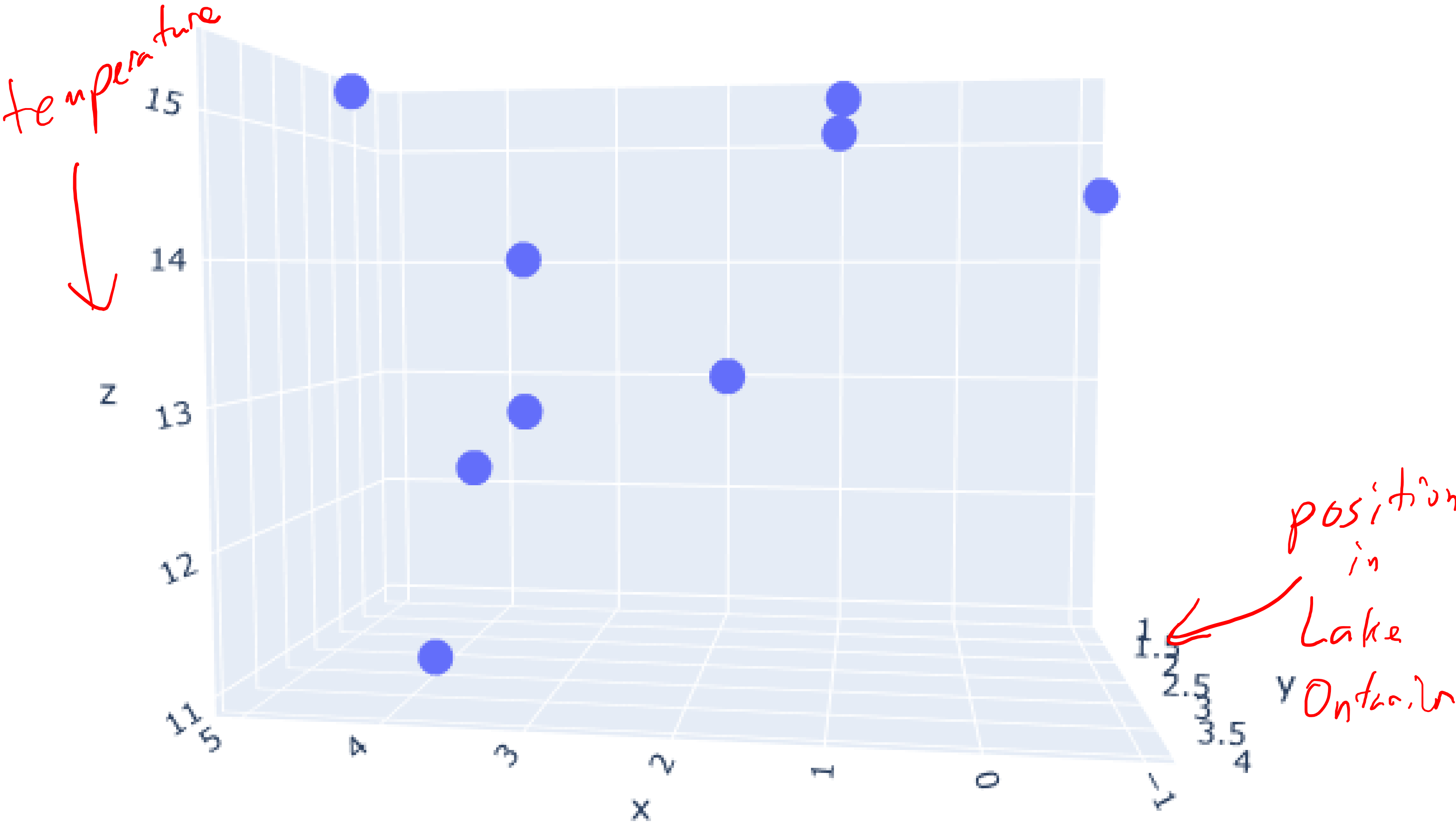


Two-variable linear regression

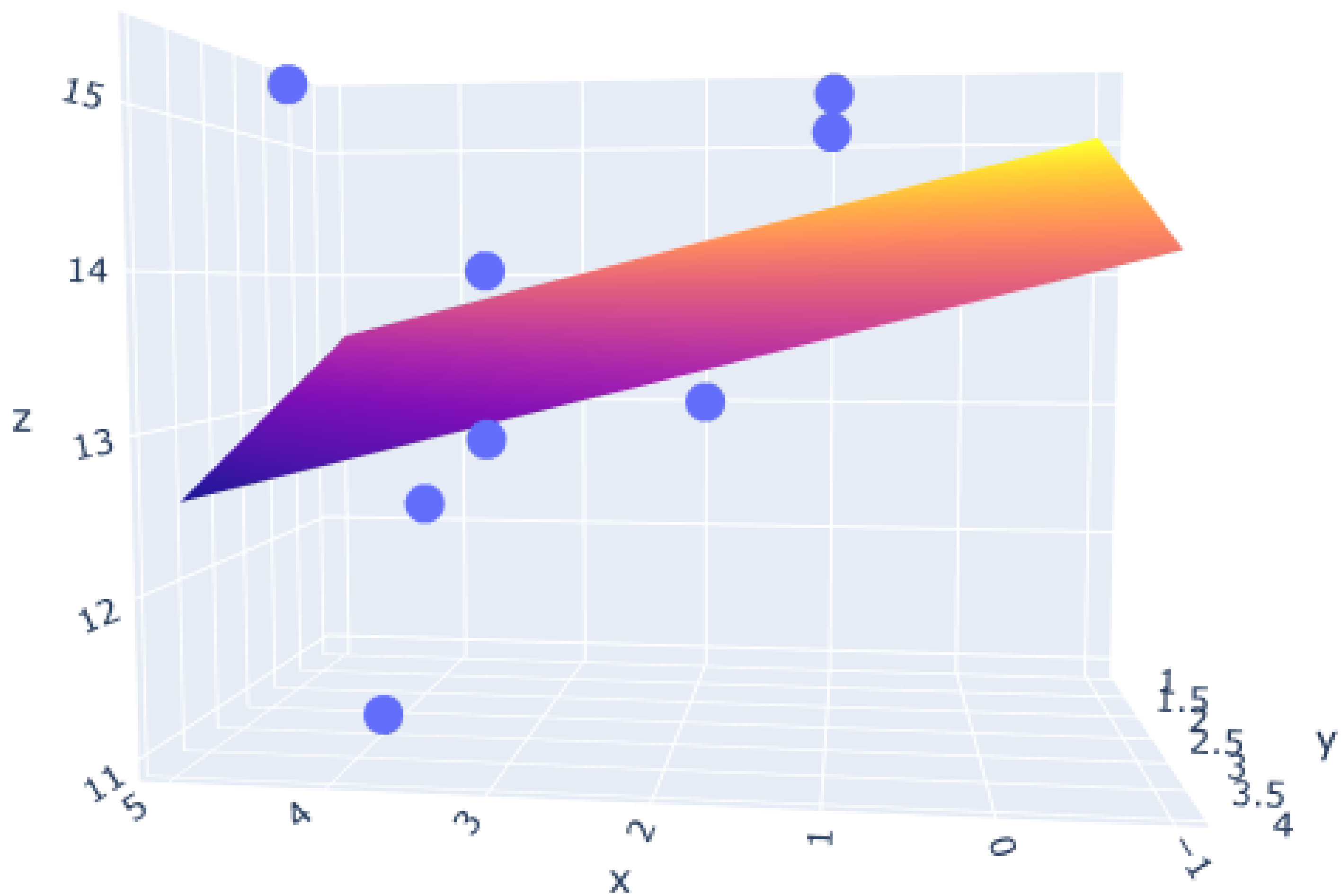
- What if we have multiple independent variables?
- Suppose we are measuring the water temperature in Lake Ontario, and want to know how the temperature varies as a function of location



3D Scatter Plot of temperatures

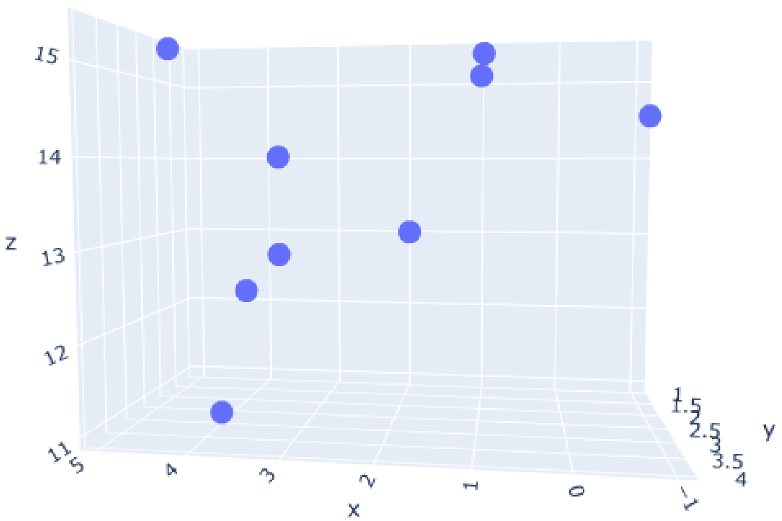


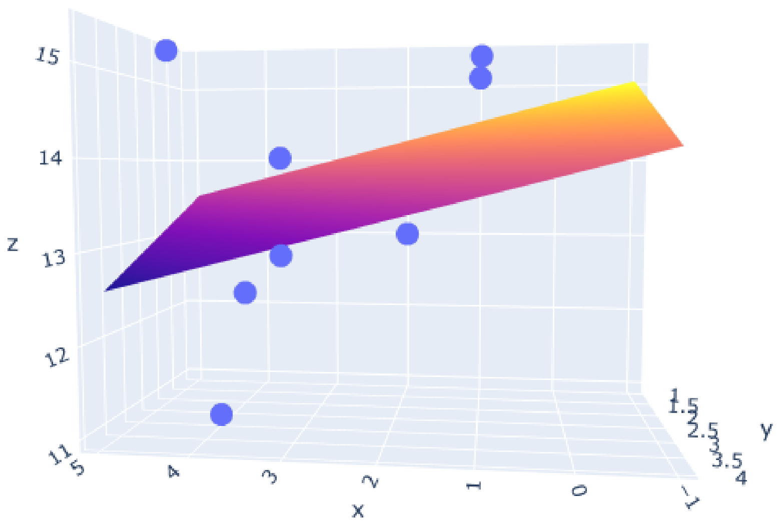
Best-fit plane



Two-variable linear regression

- Let x and y be the independent variables. Let z be the dependent variable.
  Given samples $z_1, \dots, z_n$ at values $(x_1, y_1), \dots, (x_n, y_n)$, we want the linear model
  $$f(x, y) = m_1 x + m_2 y + b$$
  such that $z_i \approx f(x_i, y_i)$, the “best-fit” plane.





Multilinear regression

- One independent variable, one dependent variable

Model: $f(x) = mx + b$

- Two independent variables, one dependent variable

Model: $f(x, y) = m_1 x + m_2 y + b$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}$

$$\Leftrightarrow \quad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + b$$

- Many independent variables, one dependent variable

Model: $f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} m_1 & \cdots & m_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b$ $\qquad f: \mathbb{R}^n \to \mathbb{R}$

- Can also have many independent variables, many dependent…

Try it out

- You are measuring the temperature of Lake Ontario as a function of location. You get the following data:

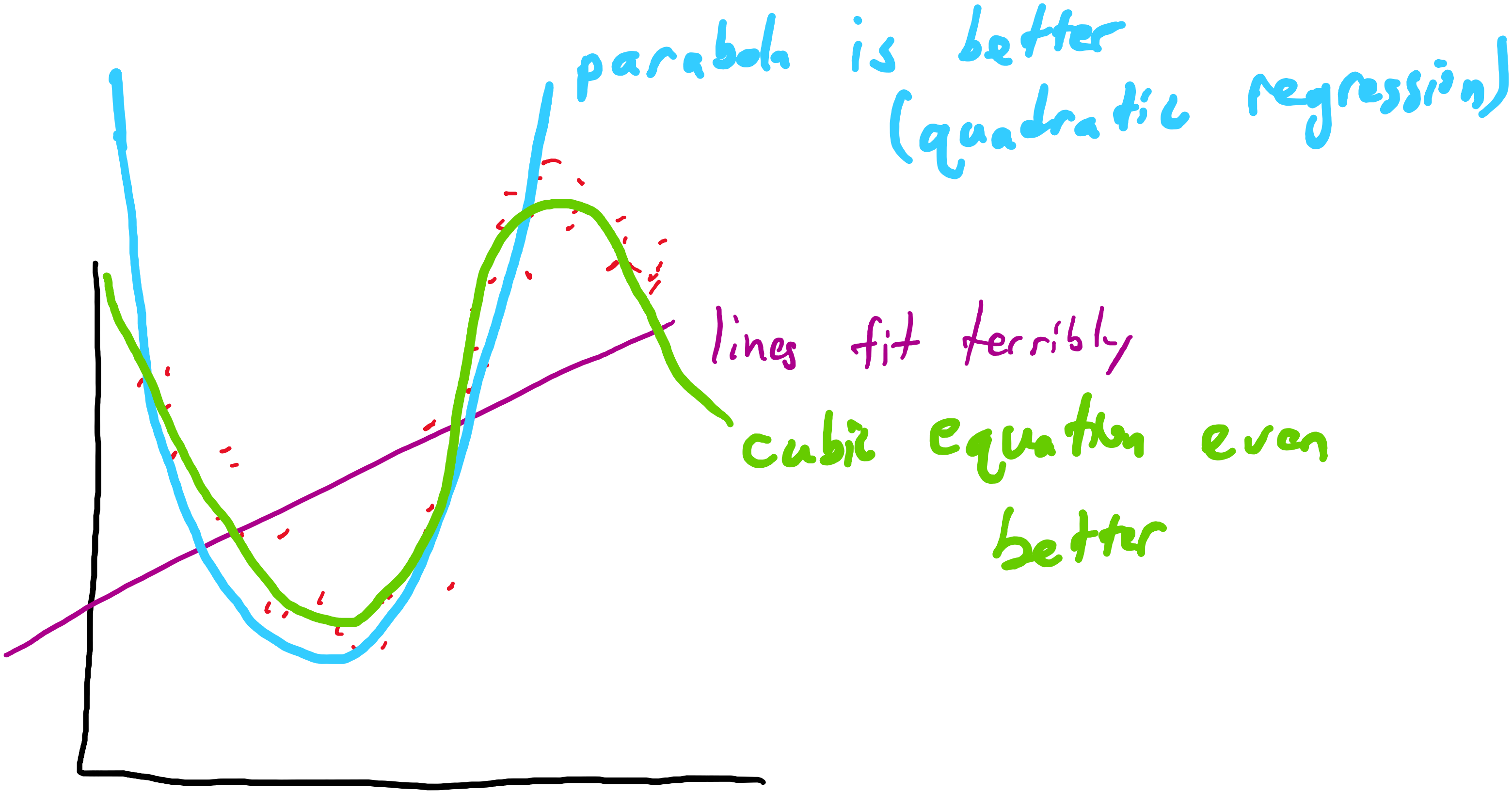
x **Longitude**	y **Latitude**	z **Temperature**
76.5 W	43.5 N	12.2
76.5 W	43.9 N	12.1
77.0 W	43.6 N	11.6
77.0 W	43.8 N	11.5
78.0 W	43.3 N	13.7
78.0 W	43.7 N	13.1
79.5 W	43.8 N	12.3
79.5 W	43.9 N	12.1

A: 12.06
B: 12.35
C: 12.54
D: 12.89
E: None of the above

- The GPS coordinates for the lake near Toronto are 43.6 N, 79.3 W. What do you predict the lake water temperature to be near Toronto?

Polynomial regression

- What if our data doesn't look linear?



Different types of regression

- Linear regression:
$$f(x) = mx + b$$
$f(x,y) = m_1 x + m_2 y + b$
- Quadratic regression:
$$f(x) = \underline{m_2 x^2 + m_1 x + b}$$
- Cubic regression:
$$f(x) = m_3 x^3 + m_2 x^2 + m_1 x + b$$
- Polynomial regression of degree n:
$$f(x) = b + \sum_{i=1}^{n} m_i x^i$$
- Exponential regression:
$$f(x) = b + m_0 e^{m_1 x}$$

Convert nonlinear to multilinear

Quadratic $\quad f(x) = m_2 x^2 + m_1 x + b$

Let $y = x^2$, $\quad f(x,y) = m_2 y + m_1 x + b$

x	$y = x^2$	$f(x)$	
1	1	0.1	
2	4	5.2	
-1	1	6.9	
4	16	4.2	
⋮	⋮	⋮	

$y = x^3$

Cubic:

$f(x) = m_3 x^3 + m_2 x^2 + m_1 x + b$

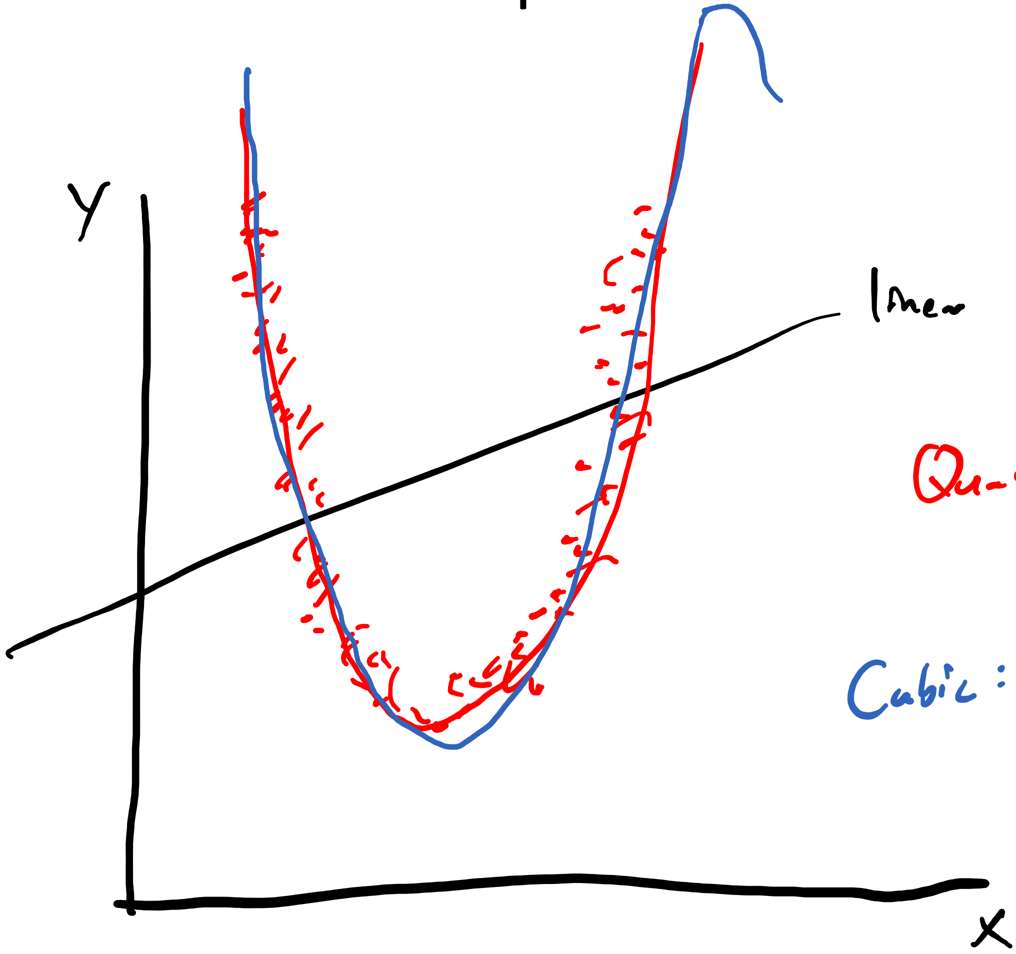
Let $x_1 = x \quad x_2 = x^2 \quad x_3 = x^3$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \overbrace{\begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}^{= m_1 x_1 + m_2 x_2 + m_3 x_3} + b$$

$= m_1 x_1 + m_2 x_2 + m_3 x_3 + b$

Intuition guess

- Linear vs. Quadratic vs Cubic: which model will have smaller Mean Square Error for the following data:



A: Linear
B: Quadratic
C: Cubic
D: Same error for All
E: None of the above

Quadratic: $f(x) = m_2 x^2 + m_1 x + b$

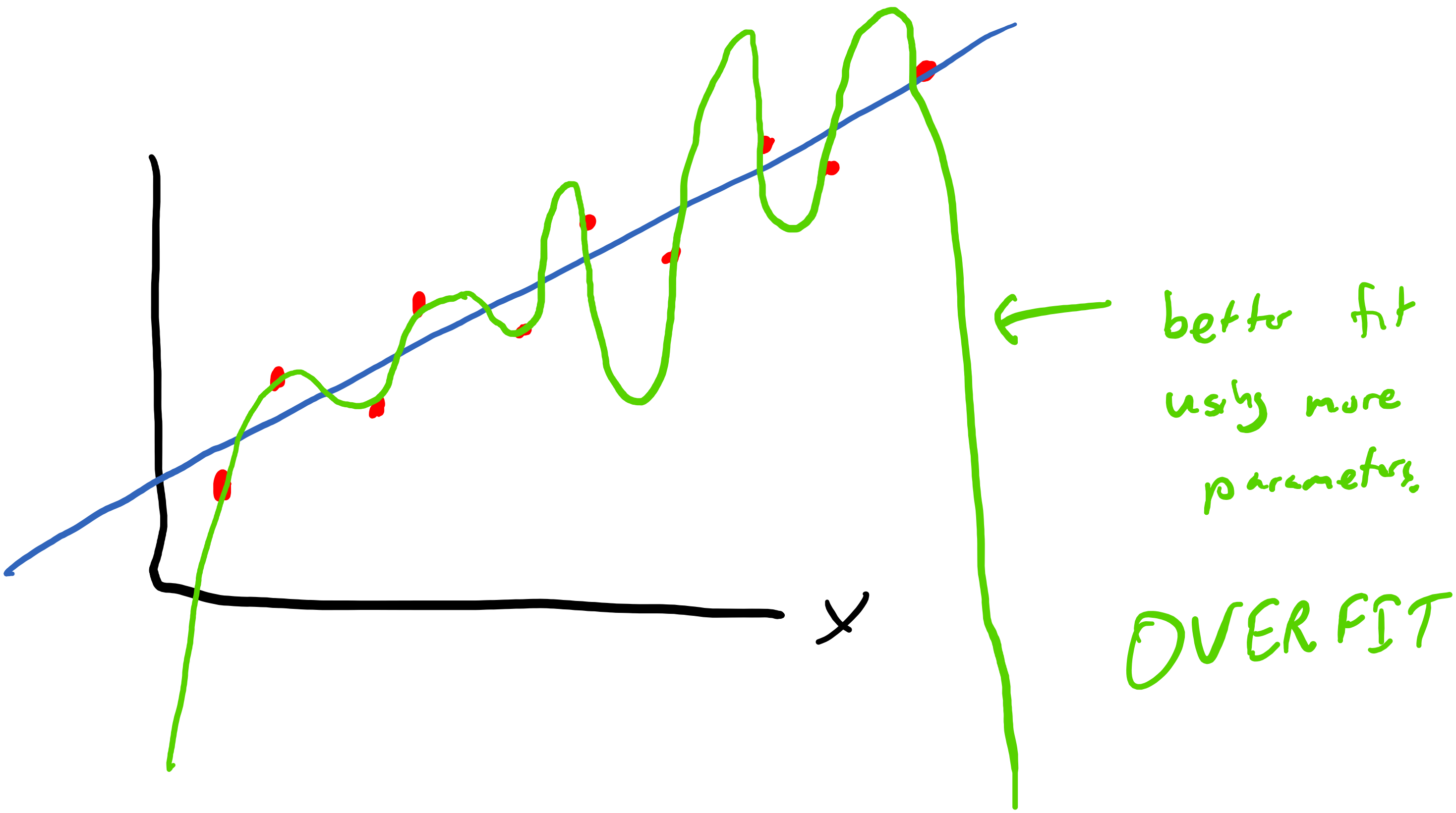
Cubic: $f(x) = m_3 x^3 + m_2 x^2 + m_1 x + b$

What if $m_3 = 0$?

For same data, cubic will always have no higher error than quadratic for "training".

Be careful about too many parameters

- The more parameters you have (e.g. in a polynomial regression), the better your mean squared error will be. on given data
- However, sometimes, you will overfit to the data.
- John von Neumann: “with four parameters, I can fit an elephant, and with five I can make him wiggle his trunk”.



Exponential regression

Exponential:

$$f(x) = c_1 e^{c_2 x}$$

$$\ln f(x) = c_2 x + \ln c_1$$

$$\ln c_1 e^{c_2 x} = \ln c_1 + \ln e^{c_2 x} = \ln c_1 + c_2 x$$

Let $z = \ln f(x)$, $m = c_2$, $b = \ln c_1$

$\Rightarrow$ $z = mx + b$

$\left(f(x) = e^{z}, \ c_2 = m, \ c_1 = e^{b} \right)$

x	f(x)

$\Rightarrow$

x	ln f(x)

Power dependencies

$$f(x) = c_1 x^{c_2}$$

$$\ln f(x) = c_2 \ln x + \ln c_1$$

Let $z = \ln f(x)$, $m = c_2$, $y = \ln x$, $b = \ln c_1$

$f(x) = e^z$, $x = e^y$ $c_1 = e^b$

$$z = my + b$$

↳ log log plot

ind ↓ ↙ dep

x	$f(x)$

→

$y = \ln x$	$z = \ln f(x)$

Power dependencies