Introduction to Ordinary Differential Equations Lecture 7a: 2023-02-27

MAT A35 – Winter 2023 – UTSC

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How do you feel after reading week?

A. Rested.
B. Excited for more quizzes!
C. Still stressed
D. Sad it's over
E. Other / reply in chat

What is a differential equation?

- An equation relates variables (e.g. x, y, z), and a solution is a set of values that makes the equation true.

$x^2 + 2y + xz = 0$ Sol: $x = 1$, $y = -1$, $z = 1$

Sol: $x = 0$, $y = 0$, $z = 0$

- A differential equation relates variables and their derivatives, and a solution is a function that makes the equation true.

$5 = \frac{dy}{dx} = y'$ Sol: $y = 5x$ Sol: $y = 5x + 1$

$x = \frac{dy}{dx}$

$x = y'$

Sol: $y = e^x$

$\frac{dy}{dx} = e^x$

Sol: $y = 100e^x$

$\frac{dy}{dx} = 100e^x$

Notational reminder

- The most unambiguous way to write derivatives is to write both variables:
  - $\frac{dy}{dx}$ is the derivative of y by the x variable.
  - $\frac{dx}{dt}$ is the derivative of x by the t variable.
- Primes/apostrophes denote the derivative by x
  - $y' = \frac{dy}{dx}$
  - $f'' = \frac{d^2 f}{dx^2}$
  - $y^{(n)} = \frac{d^n y}{dx^n}$
- Dots above a variable denote a time-derivative by t
  - $\dot{x} = \frac{dx}{dt}$
  - $\ddot{y} = \frac{d^2 y}{dt^2}$

ODEs vs PDEs

$y' = \dfrac{dy}{dx}$ ← dep var ← ind var

- Ordinary differential equations (ODEs) contain only one independent variable, so we have regular derivatives.

$F(x, y, y', y'', \ldots, y^{(n)}) = 0$ where $y = f(x)$

Ex. $x - y' = 0$, $x^2 y'' + xy' + xy^2 - 1 = 0$

- Partial differential equations (PDEs) contain multiple independent variables, so we have partial derivatives.

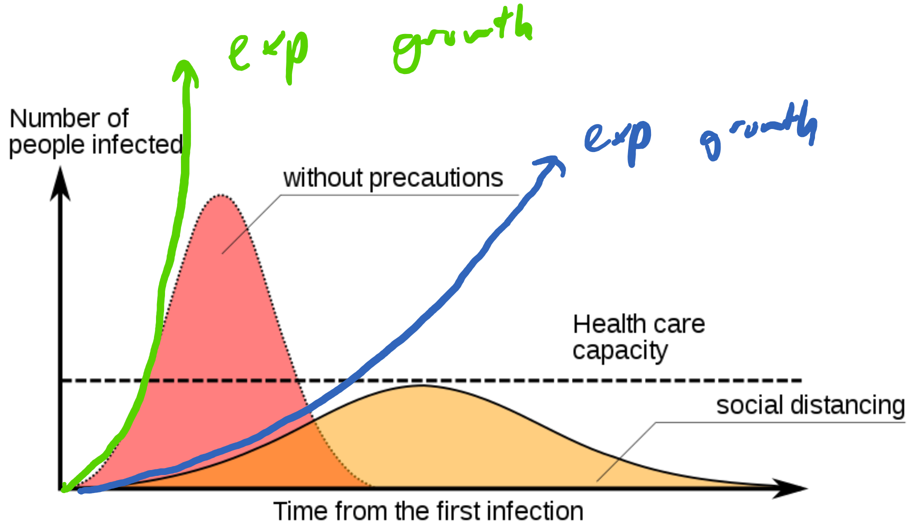
dep → $z = f(x, y)$ ← 2 ind. var

Ex. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} - z^2 y = 0$

We will not be covering PDEs in MATA35 other than to recognize them.

Example of ODE

- Early on in an epidemic, the infection rate is proportional to the number of infected individuals.
- Let $I(t)$ be the number of infected individuals at time t



https://commons.wikimedia.org/wiki/File:COVID-19_Health_care_limit.svg

$\frac{dI}{dt} = \dot{I}(t) = kI(t)$

$\dot{I}(t) = kCe^{kt} = kI(t)$

Sol: $I(t) = Ce^{kt}$, where C is a constant

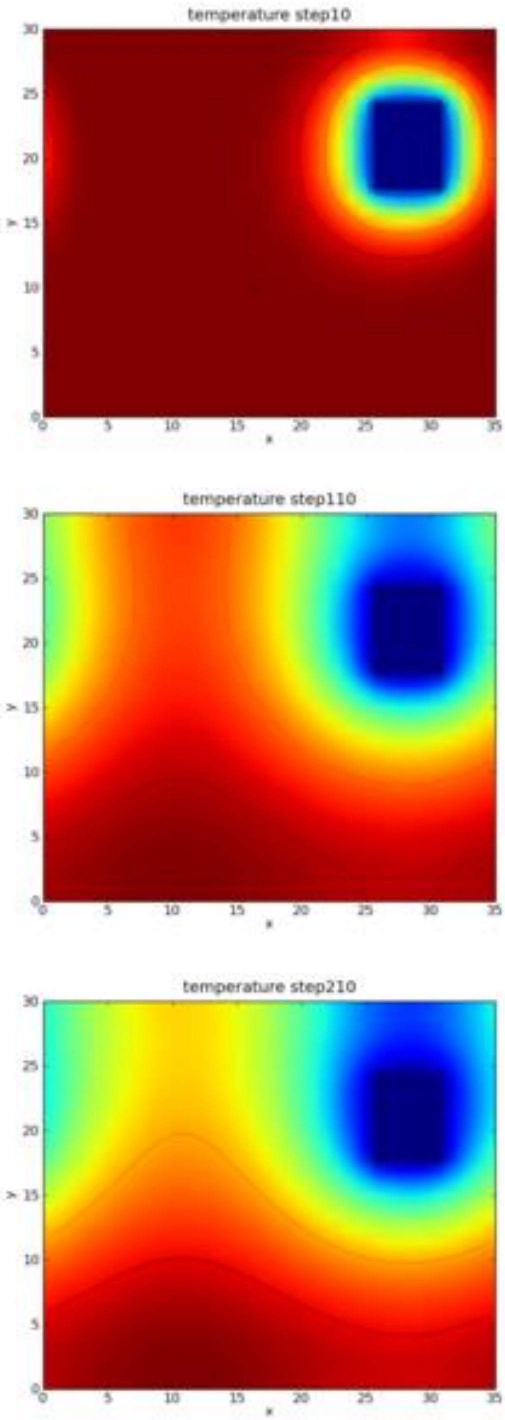
exponential growth

Example of PDE

- Heat diffusion has multiple spatia independent variables, and is normally modelled by a PDE.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$u(t, x, y, z)$ is a function of 4 variables



https://commons.wikimedia.org/wiki/File:Heat_diffusion.png

Classify the following

- $x^2 + 5x + y^2 = 5xy$ typical equation
- $x^2 + 5\underline{y'} + y^2 = 4x\underline{y'}$ ODE $y' = \frac{dy}{dx}$
- $t^2\dot{x} + 5t = x$ $\frac{dx}{dt}$ ODE
- $\left(\frac{\partial z}{\partial x}\right)^2 + y^2\frac{\partial z}{\partial y} - 5x$ None of the above because not an equation
- $\left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial y} + z^2 \underline{=} 2$ PDE

A: Typical equation
B: ODE
C: PDE
D: ???
E: None of the above

Time vs. space

- Often, in practice, ODEs describe how a system changes with time as the only independent variable (whether we call the independent variable “x” or “t”)
- Often, PDEs include how a system changes with space, and sometimes also with time, giving multiple independent variables.

Specific types of ODEs

nth order

- General form of ODEs: $F\left(x, y, y', \dots, y^{(n)}\right) = 0$
- First-order ODE: $F(x, y, y') = 0$ $F(x,y,y')=0$
- Pure-time ODE: $y' = f(x)$ ← only depends on x
- Autonomous ODE: $F\left(y, y', \dots, y^{(n)}\right) = 0$ ← don't directly dep on x
- Autonomous 1st-order: $y' = f(y)$
- Linear ODEs:
  $a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = q(x)$, ←
  where $a_i(x)$ and $q(x)$ are all functions of x.
- Linear 1st-order:
  $y' + p(x)y = q(x)$

If $a_1(x)y' + a_0(x)y = q_0(x)$

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{q_0(x)}{a_1(x)}$$

$$p(x) = \frac{a_0(x)}{a_1(x)}$$

$$q(x) = \frac{q_0(x)}{a_1(x)}$$

Try it out: classify order of ODE

- $y'' + y' + y^2 = 5$ 2nd order

$$y'(y''+1)=y$$
$$y' \cdot y'' + y' = y$$

- $y' - 1 = x^2$ 1st order

- $\underbrace{\frac{dx}{dy}}_{\text{1st}}\left(\underbrace{\frac{d^2x}{dy^2}}_{\text{2nd}} + 1\right) = y$ 2nd order

$dx \leftarrow$ dep var

$dy \leftarrow$ ind var

- $y'''' + 4y'' + 4y = x^2$ 4th order

- $\dot{x} = t^2 + 1$ 1st order

$dx \leftarrow$ dep var

$dt \leftarrow$ ind var

A: 1st order
B: 2nd order
C: 3rd order
D: 4th order
E: None of the above

Try it out: pure time/autonomous/neither

A: Autonomous
B: Pure-time
C: Both of the above
D: ???
E: None of the above

$y = f(x)$, $y \leftarrow$ dep, $x \leftarrow$ ind

$dy \leftarrow$ dep
$dx \leftarrow$ ind

- $y'' + y' + y^2 = 5$ $\quad y' = \frac{dy}{dx}$ Autonomous, no direct x-dependence
- $y' - 1 = x^2$ $\quad y' = \frac{dy}{dx}$ $\quad y' = x^2 + 1$ pure-time, y' depends only on x.
- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$ $\quad dx \leftarrow$ dep, $dy \leftarrow$ ind. None of the above because direct y-dependence and $\frac{dx}{dy}$-dependent
- $y'''' + 4y'' + 4y = x^2$ None of the above
- $\dot{x} = t^2 + 1$ $\quad dx \leftarrow$ dep, $dt \leftarrow$ ind. Pure-time
- $y' = 0$ $= f(x)$ Both.

Try it out: linear vs nonlinear

- $y'' + y' + y^2 = 5$ nonlinear
- $y' - 1 = x^2$ linear ODE because ind. var. x doesn't count
- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$ nonlinear: multiplying together two derivatives
- $y'''' + 4y'' + 4y = x^2$ linear, x^2 doesn't matter
- $\dot{x} = t^2 + 1$ linear, t^2 doesn't matter

A: Linear
B: Nonlinear
C: Both of the above
D: ???
E: None of the above