Introduction to Ordinary Differential Equations Lecture 7a: 2023-02-27

MAT A35 – Winter 2023 – UTSC

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What is a differential equation?

- An equation relates variables (e.g. x, y, z), and a solution is a set of values that makes the equation true.

- A differential equation relates variables and their derivatives, and a solution is a function that makes the equation true.

Notational reminder

- The most unambiguous way to write derivatives is to write both variables:
  - $\frac{dy}{dx}$ is the derivative of y by the x variable.
  - $\frac{dx}{dt}$ is the derivative of x by the t variable.
- Primes/apostrophes denote the derivative by x
  - $y' = \frac{dy}{dx}$
  - $f'' = \frac{d^2 f}{dx^2}$
  - $y^{(n)} = \frac{d^n y}{dx^n}$
- Dots above a variable denote a time-derivative by t
  - $\dot{x} = \frac{dx}{dt}$
  - $\ddot{y} = \frac{d^2 y}{dt^2}$

ODEs vs PDEs

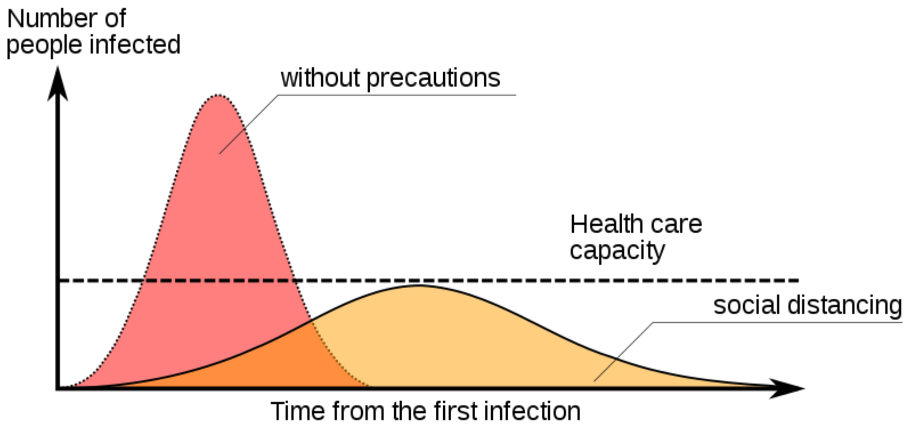
- Ordinary differential equations (ODEs) contain only one independent variable, so we have regular derivatives.

- Partial differential equations (PDEs) contain multiple independent variables, so we have partial derivatives.

We will not be covering PDEs in MATA35 other than to recognize them.

Example of ODE

- Early on in an epidemic, the infection rate is proportional to the number of infected individuals.
- Let $I(t)$ be the number of infected individuals at time t

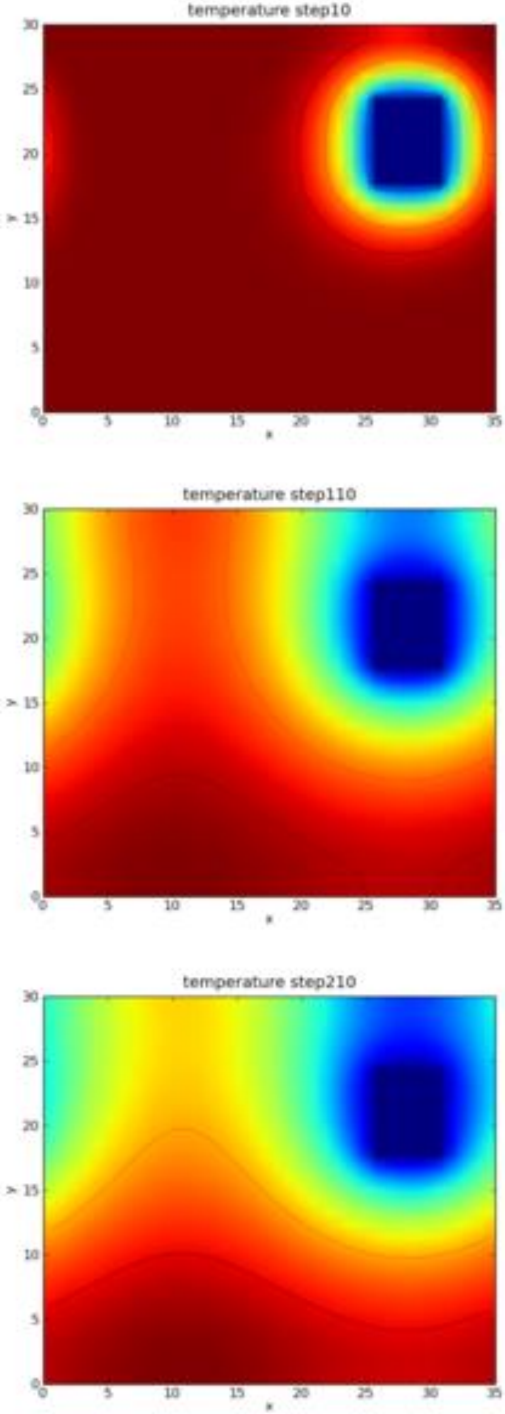


https://commons.wikimedia.org/wiki/File:COVID-19_Health_care_limit.svg

Example of PDE

- Heat diffusion has multiple spatia independent variables, and is normally modelled by a PDE.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$



https://commons.wikimedia.org/wiki/File:
Heat_diffusion.png

Classify the following

- $x^2 + 5x + y^2 = 5xy$
- $x^2 + 5y' + y^2 = 4xy'$
- $t^2\dot{x} + 5t = x$
- $\left(\frac{\partial z}{\partial x}\right)^2 + y^2\frac{\partial z}{\partial y} - 5x$
- $\left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial y} + z^2 = 2$

A: Typical equation
B: ODE
C: PDE
D: ???
E: None of the above

Time vs. space

- Often, in practice, ODEs describe how a system changes with time as the only independent variable (whether we call the independent variable “x” or “t”)
- Often, PDEs include how a system changes with space, and sometimes also with time, giving multiple independent variables.

Specific types of ODEs

- General form of ODEs: $F\left(x, y, y', \dots, y^{(n)}\right) = 0$
- First-order ODE: $F(x, y') = 0$
- Pure-time ODE: $y' = f(x)$
- Autonomous ODE: $F\left(y, y', \dots, y^{(n)}\right) = 0$
- Autonomous 1st-order: $y' = f(y)$
- Linear ODEs:
  $a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = q(x)$,
  where $a_i(x)$ and $q(x)$ are all functions of x.
- Linear 1st-order:
  $y' + p(x)y = q(x)$

Try it out: classify order of ODE

- $y'' + y' + y^2 = 5$

- $y' - 1 = x^2$

- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$

- $y'''' + 4y'' + 4y = x^2$

- $\dot{x} = t^2 + 1$

A: 1st order
B: 2nd order
C: 3rd order
D: 4th order
E: None of the above

Try it out: pure time/autonomous/neither

A: Autonomous
B: Pure-time
C: Both of the above
D: ???
E: None of the above

- $y'' + y' + y^2 = 5$
- $y' - 1 = x^2$
- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$
- $y'''' + 4y'' + 4y = x^2$
- $\dot{x} = t^2 + 1$
- $y' = 0$

Try it out: linear vs nonlinear

- $y'' + y' + y^2 = 5$
- $y' - 1 = x^2$
- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$
- $y'''' + 4y'' + 4y = x^2$
- $\dot{x} = t^2 + 1$

A: Linear
B: Nonlinear
C: Both of the above
D: ???
E: None of the above