Pure-time and separable ODEs Lecture 7b: 2023-02-27

MAT A35 – Winter 2023 – UTSC

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Pure-time ODEs: $y' = f(x)$

- Can solve via integration: $y = \int f(x)dx$

Ex.

$$y' = 2x + 5$$

$$y = \int (2x + 5)\,dx$$

$$y = x^2 + 5x + C \leftarrow \text{general sol}$$

Ex.

$$\dot{x} = \sin t - 1$$

$$\frac{dx}{dt} = \sin t - 1$$

$$x = \int dx = \int (\sin t - 1)\,dt$$

$$x = -\cos t - t + C$$

General vs Particular Solutions

- The general solution has some constant(s) in it, and covers all possible particular solutions.
- A particular solution assigns specific values to those constants.

Ex.

$$y' = 2x + 5$$

$$y = x^2 + 5x + C \leftarrow \text{gen sol.}$$

$$\left.\begin{aligned} y &= x^2 + 5x + 1 \\ y &= x^2 + 5x + 2 \end{aligned}\right\} \text{particular sols.}$$

Initial value problem

- When you specify “initial conditions”, you choose a single particular solution out of the general solution.

Ex. $y = x^2 + 5x + C$

plug in x

Initial value (IV): $y(1) = 10$ plug in y

$\Rightarrow \quad 10 = 1 + 5 + C$

$\Rightarrow \quad C = 4$

$\Rightarrow \quad y = x^2 + 5x + 4$

particular sol.

Try it out

- $\dot{x} = \sin t - 1$
- What is the solution to the initial value problem $x(0) = 0$?

$t \quad x$

$x = -\cos t - t + C$

$0 = -1 + C$

$\Rightarrow C = 1$

$x = -\cos t - t + 1$

$0 = -\cos 0 - 0 + 1$

$0 = -1 - 0 + 1$ ✓

A: $x = \sin t$
B: $x = -\cos t - t$
C: $x = -\cos t - t + 1$
D: $x = -\cos t - t + C$
E: None of the above

Separating the derivative

- Another way to think about pure-time ODEs:
- We can "split" the derivative $\frac{dy}{dx} = f(x)$ by "multiplying" by dx on both sides: $dy = f(x)dx$, and then integrate on both sides.

Ex.

$y' = 2x + 5$

$\frac{dy}{dx} = 2x + 5$

$dy = (2x + 5)dx$

$\int dy = \int (2x + 5)dx$

$y + C_1 = x^2 + 5x + C_2$

$y = x^2 + 5x + C$

"$C_2 - C_1$

Separable ODEs

- If we can split a first-order ODE so that one side has all of the dependent variable and the other side has all of the independent variable, then we can integrate on both sides.

Ex

$$2yy' + x = 0$$

$$2y\frac{dy}{dx} + x = 0$$

$$2y\frac{dy}{dx} = -x$$

$$2y\,dy = -x\,dx$$

$$\int 2y\,dy = \int -x\,dx$$

$$\underbrace{y^2 = -\frac{x^2}{2} + C}_{\text{implicit gen. sol}}$$

$$\underbrace{y = \pm\sqrt{-\frac{x^2}{2} + C}}_{\text{explicit gen sol.}}$$

Implicit vs explicit solutions

- An explicit solution to an ODE with independent variable x and dependent variable y is of the form $y = f(x)$.

- An implicit solution to an ODE with independent variable x and dependent variable y is of the form $F(x, y) = 0$.
  - Sometimes, an implicit solution is the best we can do.
  - Example on next slide.

Implicit solution initial value problem

Ex. $y' = \dfrac{2x}{y+e^{5y}}$, $y(2)=0$

$$\frac{dy}{dx} = \frac{2x}{y+e^{5y}}$$

$$\int (y+e^{5y})\,dy = \int 2x\,dx$$

$$\underbrace{\frac{y^2}{2} + \frac{e^{5y}}{5} = x^2 + C}_{\text{implicit gen sol!}}$$

$\downarrow$ IVP

$$\frac{0}{2} + \frac{1}{5} = 4 + C$$

$$C = -\frac{19}{5}$$

$$\underbrace{\frac{y^2}{2} + \frac{e^{5y}}{5} = x^2 - \frac{19}{5}}_{\text{implicit particular solution}}$$

Try it out: is the following separable?

$\frac{dy}{dx}$

- $y' - y^2 x \sin x^2 = 0$ YES

$$\frac{dy}{dx} = y^2 x \sin x^2$$

$$\frac{dy}{y^2} = x \sin x^2 \, dx$$

- $\dot{x}x^2 + t^2 e^t = 4$ YES

$$x^2 \frac{dx}{dt} = 4 - t^2 e^t$$

$$x^2 dx = (4 - t^2 e^t) dt$$

- $\dot{x}t^2 + x^2 e^x = 4$ YES

$$t^2 \cdot \frac{dx}{dt} + x^2 e^x = 4$$

$$t^2 \frac{dx}{dt} = 4 - x^2 e^x$$

$$\frac{dx}{4 - x^2 e^x} = \frac{dt}{t^2}$$

- $y' = xy$ YES

$$\frac{dy}{dx} = xy \qquad \frac{dy}{dx} = xy \Rightarrow \int \frac{dy}{y} = \int x \, dx \qquad \ln|y| = \frac{1}{2}x^2 + C$$

- $y' = \sin(xy)$ NO

A: Separable
B: Not separable
C: Cannot tell
D: ???
E: None of the above

Example: $y' = xy$, where $y(1) = 1$

$\frac{dy}{dx} = xy$

$\int \frac{dy}{y} = \int x\,dx$

$\ln|y| = \frac{1}{2}x^2 + C$

IVP

$x = 1$

$y = 1$

$\ln|1| = \frac{1}{2} + C$

$0 = \frac{1}{2} + C$

$C = -\frac{1}{2}$

$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}$

$y = e^{\left(\frac{1}{2}x^2 - \frac{1}{2}\right)}$

Try it out: $y' = 3x^2e^{2y}$

$$\frac{dy}{dx} = 3x^2 e^{2y}$$

$$\frac{dy}{e^{2y}} = 3x^2\,dx$$

$$\int e^{-2y}\,dy = \int 3x^2\,dx$$

$$-\frac{1}{2}e^{-2y} = x^3 + C$$

$$e^{-2y} = -2x^3 + C$$

$$-2y = \ln|-2x^3 + C|$$

$$y = -\frac{1}{2}\ln|-2x^3 + C|$$

A: $y = -\frac{1}{2}\ln|-2x^3 + C|$

B: $e^{-2y} = -2x^3 + C$

C: $-\frac{1}{2}e^{-2y} = x^3 + C$

D: All of the above

E: None of the above