Exact differentials and integrating factors Lecture 7c: 2023-02-27

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

Existence-Uniqueness Theorem

- Consider the 1st order linear ODE initial value problem
$$y' + p(x)y = q(x), \qquad\qquad y(x_0) = y_0$$
- If p and q are continuous functions on an interval I containing x_0, then there exists a unique solution to the IVP for every point in I.

- In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

Differentials

- Differentials dx and dy are the intuition behind $\frac{dy}{dx}$, and can be thought of as infinitesimal changes along the x- or y-axes.

Ex.

$y' = x^2$

$\frac{dy}{dx} = x^2$

$\int dy = \int x^2 dx$

$y = \frac{1}{3}x^3 + C$

Let $f(x) = \frac{1}{3}x^3$

$d[f(x)] = d\left[\frac{1}{3}x^3\right]$

$df = d\left[\frac{1}{3}x^3\right] = x^2 dx$

Differentials of multi-variable functions

- Let $z = f(x, y)$ be a function of both x and y.
- Recall that the gradient $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ gives the partial derivative in the $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ direction by $\nabla f \cdot u$.
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- i.e. $dz = \nabla f \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$, where ∇f is the Gradient.

Example of total differential

$d[x] = dx$

$d[x^2] = 2x\,dx$

Ex

$$z = x^2 + 2xy + y^4$$

$$dz = \frac{\partial}{dx}[x^2 + 2xy + y^4]\,dx + \frac{\partial}{\partial y}[x^2 + 2xy + y^4]\,dy$$

$$dz = (2x + 2y)\,dx + (2x + 4y^3)\,dy$$

$$dz = 2x\,dx + 2y\,dx + 2x\,dy + 4y^3\,dy$$

Ex,

$$f(x,y) = \sin 2x + y^2 e^x$$

$$df = 2\cos 2x\,dx + y^2 e^x\,dx + 2ye^x\,dy$$

Ex.

$$z = x^2 y^2$$

$$dz = 2xy^2\,dx + 2x^2 y\,dy$$

Try it out: compute the total differential

- $f(x,y) = x^2 + e^y \sin x$

$df = (2x + e^y \cos x)\,dx + e^y \sin x \, dy$

A: $(2x + e^y \cos x)dx + e^y \sin x \, dy$
B: $x^2 dx + e^y \sin x \, dy$
C: $2x dx + e^y \cos x \, dx$
D: $2x dy + e^y \sin x \, dy$
E: None of the above

- $z = \dfrac{5x^2}{y-1} + 1$

$dz = \frac{\partial}{\partial x}\left[\frac{5x^2}{y-1} + 1\right] dx + \frac{\partial}{\partial y}\left[\frac{5x^2}{y-1} + 1\right] dy$

$dz = \frac{10x}{y-1} dx - \frac{5x^2}{(y-1)^2} dy$

A: $\frac{5x^2}{y-1} dx - \frac{5x^2}{(y-1)^2} dy$
B: $\frac{10x}{y-1} dx - \frac{5x^2}{y-1} dy$
C: $\frac{10x}{y-1} dx + \frac{5x^2}{(y-1)^2} dy$
D: $\frac{10x}{y-1} dx - \frac{5x^2}{(y-1)^2} dy$
E: None of the above

Reversing a total differential

- $dz = (2x + 2y)dx + (2x + 4y^3)dy$
- Solve for $\frac{\partial z}{\partial x} = 2x + 2y$ and $\frac{\partial z}{\partial y} = 2x + 4y^3$

$z = \int (2x + 2y)\,dx$

$z = x^2 + 2xy + F(y)$

$z = \int (2x + 4y^3)\,dy$

$z = 2xy + y^4 + G(x)$

$z = x^2 + 2xy + y^4 + C$

Try it out: Find z such that

- $dz = \left(2xy \cdot e^{x^2y}\right)dx + \left(x^2 \cdot e^{x^2y} + 5\right)dy$

$z = \int 2xy\, e^{x^2y}\, dx$

$z = \int (x^2 e^{x^2y} + 5)\, dy$

$z = e^{x^2y} + F(y)$

$z = e^{x^2y} + 5y + G(x)$

$z = \underline{e^{x^2y} + 5y} + C$

$C = 0 \Rightarrow z = e^{x^2y} + 5y$

A: $e^{x^2y} + 5y$
B: $x^2ye^{x^2y} + 5xy$
C: $x^2ye^{x^2y} + 5y$
D: $e^{2xy} + 5y$
E: None of the above

Sometimes, reversing fails

- $dz = y^2dx + x^2dy$

$z = \int y^2 dx$

$z = \int x^2 dy$

$z = xy^2 + \boxed{F(y)}$ $\neq$ $z = x^2y + \boxed{G(x)}$

"NOT EXACT"

Exact differential

- A differential
$$dz = P(x,y)dx + Q(x,y)dy$$
is an exact differential if there exists a function $f(x,y)$ such that $P(x,y) = \frac{\partial f}{\partial x}$ and $Q(x,y) = \frac{\partial f}{\partial y}$. Then $z = f(x,y)$.
- In other words, an exact differential is any differential that is the total differential of some function.
- An inexact differential is a differential we write down that is not the total differential of any function.

Differential test for exactness

- One way to test for exactness is to try to reverse the differential; this will always work, but involves a lot of integration.
- There is a faster test that only involves differentiation.
- Recall that for most nice functions $\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}$.
- Therefore, quick way to see if a differential

$$\underline{dz = P(x,y)dx + Q(x,y)dy}$$

is exact is to check if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \quad = \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

Example

Ex.

$$dz = \frac{10x}{y-1}dx - \frac{5x^2}{(y-1)^2}dy$$

$$\underbrace{\frac{10x}{y-1}}_{P(x,y)} \qquad \underbrace{\frac{5x^2}{(y-1)^2}}_{Q(x,y)}$$

$$\frac{\partial P}{\partial y} = -\frac{10x}{(y-1)^2} = \frac{\partial Q}{\partial x} = \frac{-10x}{(y-1)^2}$$

Exact

Ex.

$$dz = (5x^2+1)dx + xy^2dy$$

$$\underbrace{(5x^2+1)}_{P(x,y)} \qquad \underbrace{xy^2}_{Q(x,y)}$$

$$\frac{\partial P}{\partial y} = 0 \neq \frac{\partial Q}{\partial x} = y^2$$

Not exact

Try it out: exact or inexact?

- $dz = xdx + ydy$

$\frac{\partial x}{\partial y} = 0 \quad \frac{\partial y}{\partial x} = 0 \quad \checkmark$ Exact, $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$

- $dz = ydx + xdy$

$\frac{\partial y}{\partial y} = 1 \quad \frac{\partial x}{\partial x} = 1$ Exact $z = xy$

- $dz = xdx + y^2dy$

$\frac{\partial x}{\partial y} = 0 \quad \frac{\partial}{\partial x}[y^2] = 0$ Exact. $z = \frac{1}{2}x^2 + \frac{1}{3}y^3$

- $dz = y^2dx + xdy$

$\frac{\partial}{\partial y}(y^2) = 2y \quad \frac{\partial}{\partial x}(x) = 1$ $\boxed{\text{Inexact.}}$

- $dz = (x + y)dx + (x + y)dy$

Exact

$\frac{\partial}{\partial y}(x+y) = 1 \quad \frac{\partial}{\partial x}(x+y) = 1$

$z = \frac{1}{2}(x+y)^2$

A: exact
B: inexact
C: both exact and inexact
D: ???
E: None of the above

Solving exact differential equations

How many assessments do you have this wee :?
(not counting M. 35)

A: 1
B: 2
C: 3
D: 4
E: 0 :)

$$\frac{dy}{dx} = \frac{-2xy}{x^2+1}$$

$$(x^2+1)\,dy = -2xy\,dx$$

$$\underbrace{2xy}_{P(x,y)}\,dx + \underbrace{(x^2+1)}_{Q(x,y)}\,dy = 0$$

$$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x \Rightarrow \text{exact}$$

$$\int (x^2+1)\,dy = x^2y + y + C_1(x)$$

$$\int 2xy\,dx = x^2y + C_2(y)$$

Guess: $f(x,y) = x^2y + y$

$$df = 2xy\,dx + (x^2+1)\,dy$$

$$\int df = f + C$$

$$\Rightarrow x^2y + y + C = 0 \Leftarrow \text{implicit form}$$

$$\Rightarrow y = \frac{-C}{x^2+1} \Leftarrow \text{explicit form}$$

Integrating factors

- Sometimes, we can find an "integrating factor" $I(x)$ to multiply by both sides of an inexact ODE to make it an exact ODE.

Ex. $xy' + 2y = 0$

$x\frac{dy}{dx} + 2y = 0$

$2y\,dx + x\,dy = 0$

$\frac{\partial}{\partial y}[2y] = 2 \qquad \frac{\partial}{\partial x}[x] = 1$

so inexact

Let $I(x) = x$

Multiply by $I(x)$

$2xy\,dx + x^2\,dy = 0$

$\frac{\partial}{\partial y}[2xy] = 2x \qquad \frac{\partial}{\partial x}[x^2] = 2x$

$\Rightarrow$ exact

Let $f(x,y) = x^2 y$

$\Rightarrow x^2 y = 0 + C$

$\Rightarrow y = \frac{C}{x^2}$.

Try it out:

$$\frac{\partial}{\partial y}\left[\frac{1}{x^2}\right]=0 \qquad \frac{\partial}{\partial x}\left[\frac{1}{xy}\right]=-\frac{1}{x^2y}$$

- Given the inexact differential equation

multiply by x

$$2dx = \frac{1}{x^2}dx + \frac{1}{xy}dy$$

- Which of the following is an integrating factor $I(x)$?

$$2x\,dx = \frac{1}{x}dx + \frac{1}{y}dy$$

$$\frac{\partial}{\partial y}\left[\frac{1}{x}\right]=0 \qquad \frac{\partial}{\partial x}\left[\frac{1}{y}\right]=0 \qquad \text{exact.}$$

$$2x^2\,dx = \cancel{d}x + \frac{x}{y}dy$$

$$\frac{\partial}{\partial y}[1]=0 \qquad \frac{\partial}{\partial x}\left[\frac{x}{y}\right]=\frac{1}{y} \qquad \text{not exact}$$

A: x
B: x^2
C: xy
D: All of the above
E: None of the above

Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.

- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known.

Integrating Factor for linear 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

which is equivalent to

$$dy + dx[p(x)y] = q(x)$$

- Then the integrating factor is

$$e^{\int p(x)dx}$$

memorize

Ex

$$xy' + 2y = 5x^3$$

$$\boxed{y' + \frac{2}{x}y = 5x^2}$$

$$dy + \frac{2}{x}y\,dx = 5x^2dx$$

$$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|}$$

$$= e^{\ln x^2} = x^2$$

multiply

$$\underbrace{dx[2xy] + dy[x^2]}_{\text{exact}} = 5x^4dx$$

General solution for 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:
$$y' + p(x)y = q(x)$$
- The general solution can be found by:
  - Determining the integrating factor $I(x) = e^{\int p(x)dx}$
  - Multiply both sides by $I(x)$: $y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
  - Multiply both sides by dx: $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
  - The left hand side is the total differential $d[I(x)y]$
  - So we can integrate both sides to get $I(x)\text{y} = \int q(x)I(x)dx$
  - Then $y = \frac{1}{I(x)}\left[\int q(x)I(x)dx + C\right]$
- In a single, ugly, long equation:
$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}dx + C\right]$$

Try it out

- $\underbrace{y' - 3x^2}_{p(x)}y = \underbrace{x^2}_{q(x)}$

$$I(x) = e^{\int -3x^2\,dx} = e^{-x^3}$$

$$-3x^2y\,dx + dy = x^2\,dx$$

$$\int \left[-3x^2 y e^{-x^3}\,dx + e^{-x^3}\,dy \right] = \underbrace{x^2 e^{-x^3}\,dx}$$

$$\int x^2 e^{-x^3}\,dx$$

$$u = x^3$$

$$du = 3x^2\,dx$$

$$= \frac{1}{3}\int e^{-u}\,du$$

$$= -\frac{1}{3}e^{-u} + C$$

$$= -\frac{1}{3}e^{-x^3} + C$$

$$y e^{-x^3} = -\frac{1}{3}e^{-x^3} + C$$

multiply by $\underline{e^{x^3}}$

$$y = -\frac{1}{3} + Ce^{x^3}$$

$$e^{-x^3} \cdot e^{x^3} = e^{-x^3+x^3} = e^0 = 1$$

A: $-\frac{1}{3} + e^{x^3} + C$

B: $-\frac{1}{3}e^{x^3} + C$

C: $-\frac{1}{3}e^{x^3+C}$

D: $-\frac{1}{3} + Ce^{x^3}$

E: None of the above