Exact differentials and integrating factors Lecture 7c: 2023-02-27

MAT A35 – Winter 2023 – UTSC

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Existence-Uniqueness Theorem

- Consider the 1st order linear ODE initial value problem
$$y' + p(x)y = q(x), \qquad y(x_0) = y_0$$
- If p and q are continuous functions on an interval I containing x_0, then there exists a unique solution to the IVP for every point in I.

- In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

Differentials

- Differentials dx and dy are the intuition behind $\frac{dy}{dx}$, and can be thought of as infinitesimal changes along the x- or y-axes.

Differentials of multi-variable functions

- Let $\mathrm{z} = f(x, y)$ be a function of both x and y.
- Recall that the gradient $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ gives the partial derivative in the $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ direction by $\nabla f \cdot u$.
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- i.e. $dz = \nabla f \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$, where ∇f is the Gradient.

Example of total differential

Try it out: compute the total differential

- $f(x, y) = x^2 + e^y \sin x$

A: $(2x + e^y \cos x)dx + e^y \sin x\, dy$
B: $x^2 dx + e^y \sin x\, dy$
C: $2x dx + e^y \cos x\, dx$
D: $2x dy + e^y \sin x\, dy$
E: None of the above

- $z = \frac{5x^2}{y-1} + 1$

A: $\frac{5x^2}{y-1}dx - \frac{5x^2}{(y-1)^2}dy$
B: $\frac{10x}{y-1}dx - \frac{5x^2}{y-1}dy$
C: $\frac{10x}{y-1}dx + \frac{5x^2}{(y-1)^2}dy$
D: $\frac{10x}{y-1}dx - \frac{5x^2}{(y-1)^2}dy$
E: None of the above

Reversing a total differential

- $dz = (2x + 2y)dx + (2x + 4y^3)dy$
- Solve for $\frac{\partial z}{\partial x} = 2x + 2y$ and $\frac{\partial z}{\partial y} = 2x + 4y^3$

Try it out: Find z such that

- $dz = \left(2xy \cdot e^{x^2y}\right)dx + \left(x^2 \cdot e^{x^2y} + 5\right)dy$

A: $e^{x^2y} + 5y$
B: $x^2ye^{x^2y} + 5xy$
C: $x^2ye^{x^2y} + 5y$
D: $e^{2xy} + 5y$
E: None of the above

Sometimes, reversing fails

- $dz = y^2 dx + x^2 dy$

Exact differential

- A differential

$$dz = P(x,y)dx + Q(x,y)dy$$

is an exact differential if there exists a function $f(x,y)$ such that $P(x,y) = \frac{\partial f}{\partial x}$ and $Q(x,y) = \frac{\partial f}{\partial y}$. Then $z = f(x,y)$.

- In other words, an exact differential is any differential that is the total differential of some function.
- An inexact differential is a differential we write down that is not the total differential of any function.

Differential test for exactness

- One way to test for exactness is to try to reverse the differential; this will always work, but involves a lot of integration.
- There is a faster test that only involves differentiation.
- Recall that for most nice functions $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.
- Therefore, quick way to see if a differential

$$dz = P(x,y)dx + Q(x,y)dy$$

is exact is to check if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Example

Try it out: exact or inexact?

- $dz = xdx + ydy$
- $dz = ydx + xdy$
- $dz = xdx + y^2dy$
- $dz = y^2dx + xdy$
- $dz = (x + y)dx + (x + y)dy$

A: exact
B: inexact
C: both exact and inexact
D: ???
E: None of the above

Solving exact differential equations

Integrating factors

- Sometimes, we can find an “integrating factor” $I(x)$ to multiply by both sides of an inexact ODE to make it an exact ODE.

Try it out:

- Given the inexact differential equation

$$2dx = \frac{1}{x^2}dx + \frac{1}{xy}dy$$

- Which of the following is an integrating factor $I(x)$?

A: x
B: x^2
C: xy
D: All of the above
E: None of the above

Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.

- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known.

Integrating Factor for linear 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

which is equivalent to

$$dy + dx[p(x)y] = q(x)$$

- Then the integrating factor is

$$e^{\int p(x)dx}$$

General solution for 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

- The general solution can be found by:
  - Determining the integrating factor $I(x) = e^{\int p(x)dx}$
  - Multiply both sides by $I(x)$: $y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
  - Multiply both sides by dx: $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
  - The left hand side is the total differential $d[I(x)y]$
  - So we can integrate both sides to get $I(x)\mathrm{y} = \int q(x)I(x)dx$
  - Then $y = \frac{1}{I(x)}\left[\int q(x)I(x)dx + C\right]$
- In a single, ugly, long equation:

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}dx + C\right]$$

Try it out

- $y' - 3x^2y = x^2$

A: $-\frac{1}{3} + e^{x^3} + C$
B: $-\frac{1}{3}e^{x^3} + C$
C: $-\frac{1}{3}e^{x^3+C}$
D: $-\frac{1}{3} + Ce^{x^3}$
E: None of the above