Direction fields, autonomous ODEs, and the phase line
Lecture 7d: 2023-03-02

MAT A35 – Winter 2023 – UTSC

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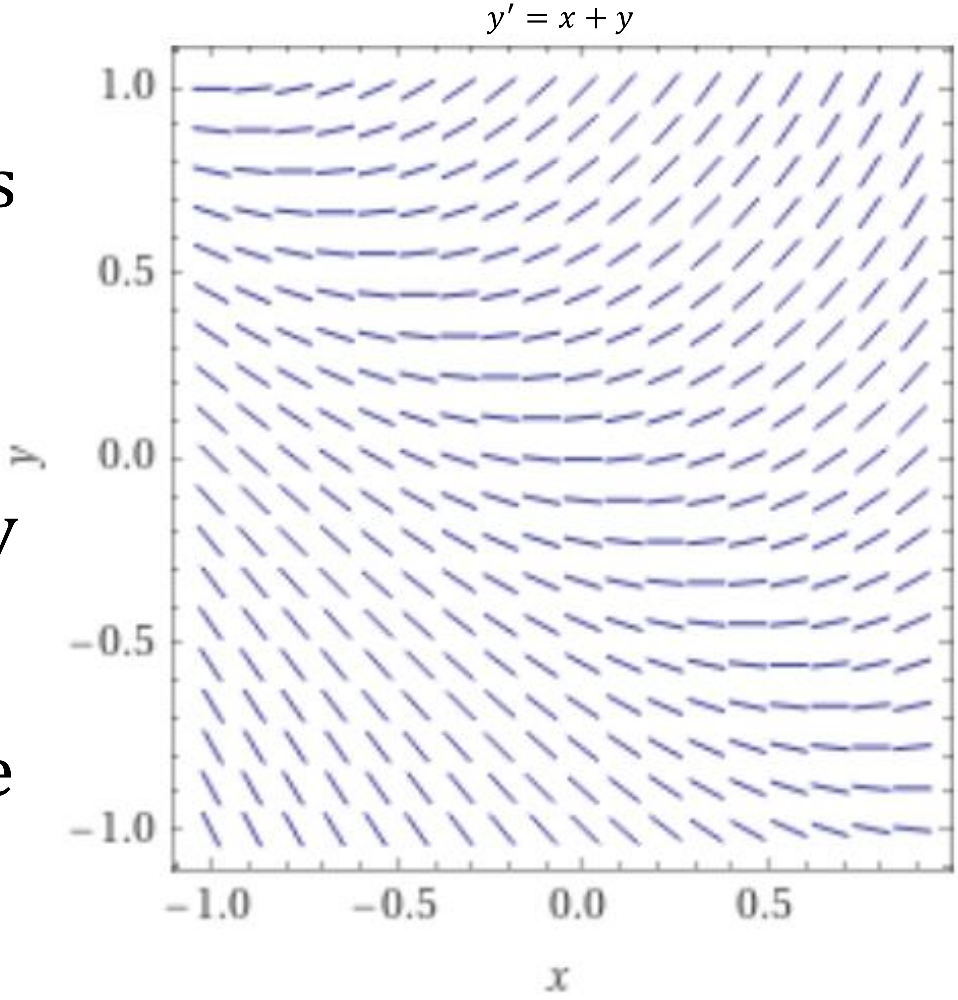
1^{st}-order ODEs and slopes of solutions

- We can write a 1^{st}-order ODE as $y' = f(x, y)$
- Recall that the derivative can be thought of as the slope of a solution.

Direction field

- A direction field graphs out the slopes of all solutions going through a point.
- We can visualize different solutions by drawing trajectory curves that are always tangent to the direction field.



https://www.wolframalpha.com/input/?i=slope+field+of+y%27%3Dx%2By

Autonomous ODEs

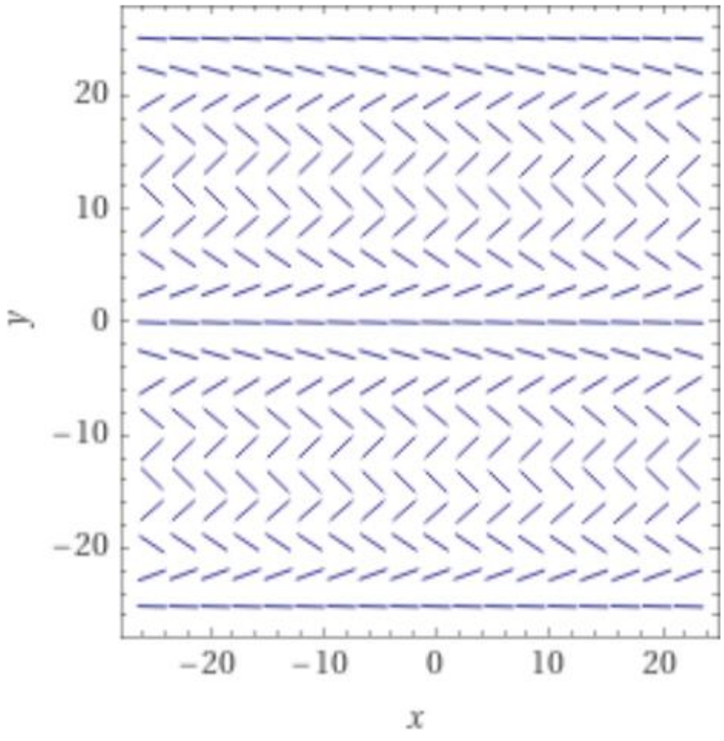
- Recall that an autonomous ODE is one that does not have an explicit dependence on the independent variable (e.g. time).
- A first-order autonomous ODE can be rewritten in the form:

$$y' = f(y)$$

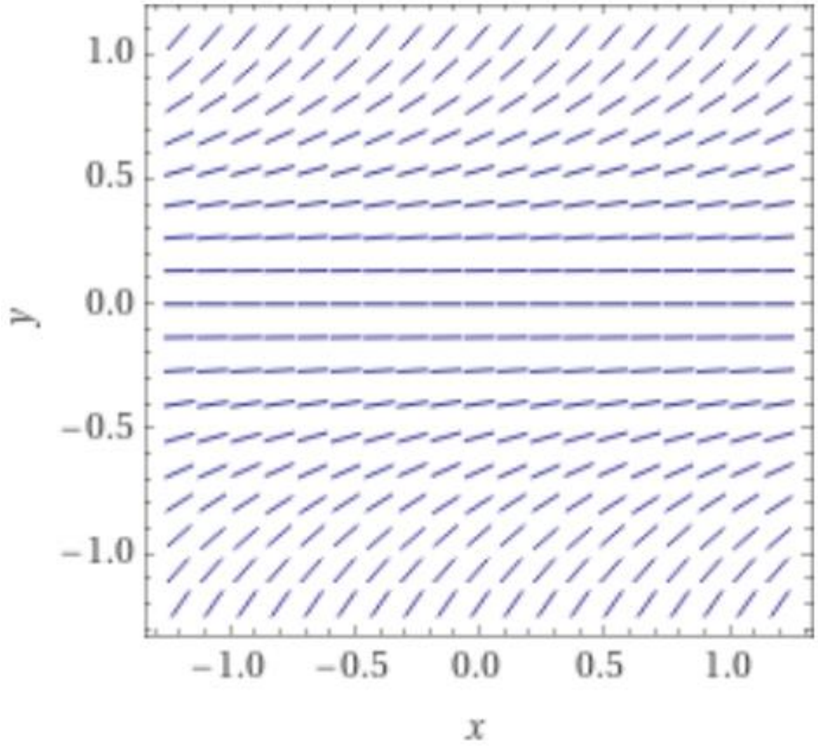
Direction fields of autonomous ODEs

- Notice that if $y' = f(y)$, then the slope has no x-dependence.

$y' = \sin y$



$y' = y^2$

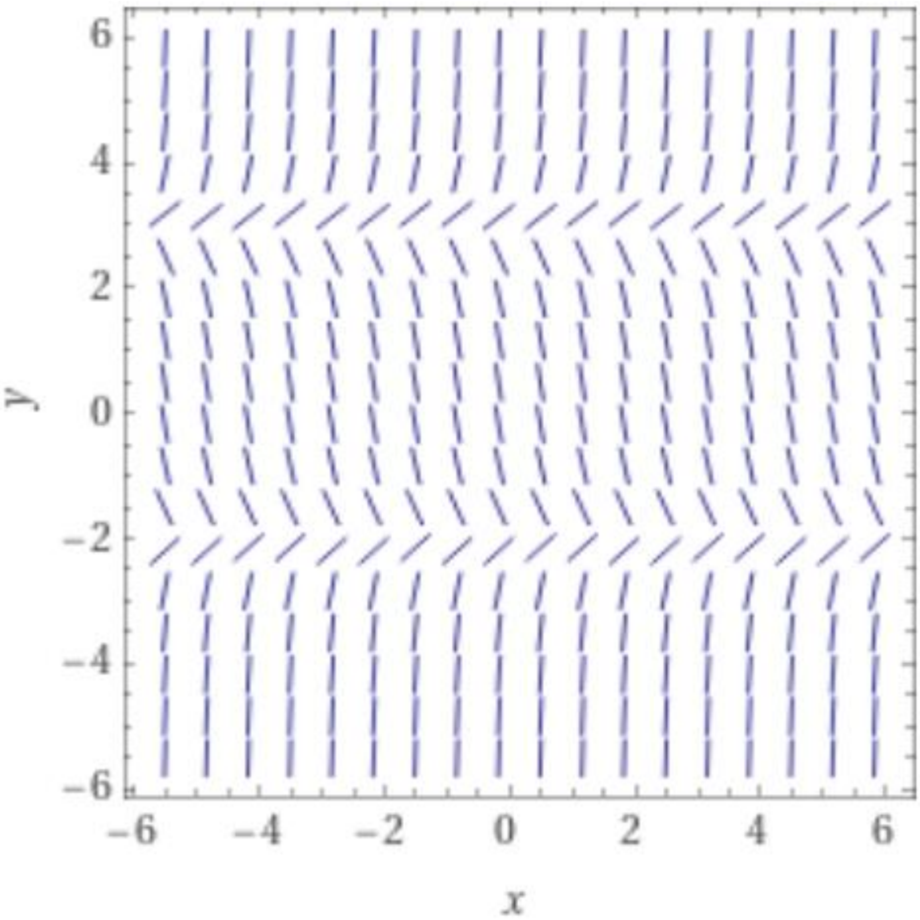


Equilibrium values

- An equilibrium value of the autonomous ODE $y' = f(y)$ is a constant solution $y = c$.
- We can solve for equilibrium values by setting $y' = 0$.

Try it out: find the equilibrium values

- $y' = y^2 - y - 6$



A: -2
B: 3
C: All of the above
D: ???
E: None of the above

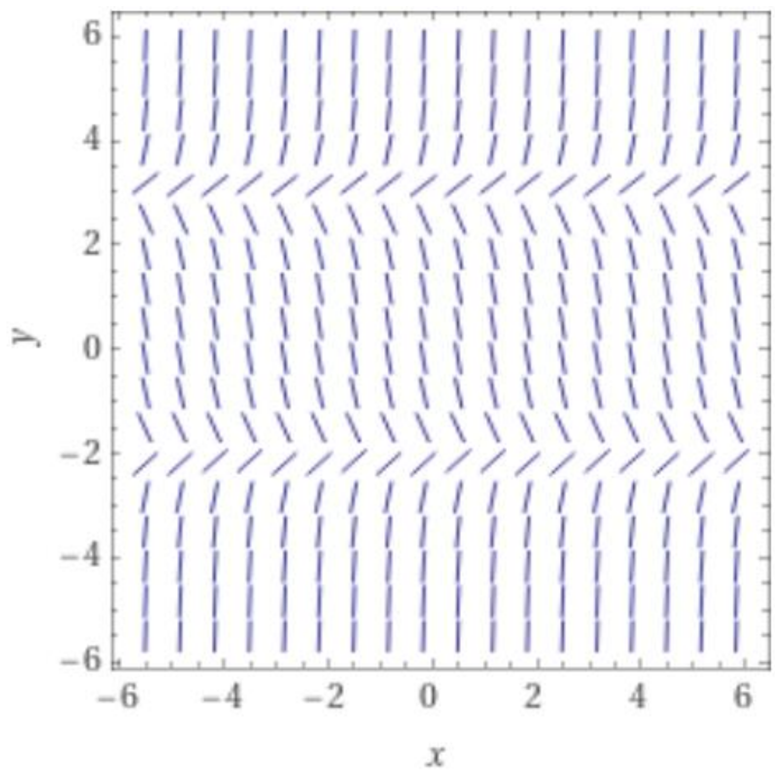
Stability of equilibrium values

- Consider an equilibrium value c of $y' = f(y)$, and an initial value $y(0) = y_0$, where $y_0 \approx c$, but $y_0 \neq c$. Then c is
  - Unstable if y diverges from c as time $x \to \infty$
  - Asymptotically stable if $y(x) \to c$ as $x \to \infty$
  - Semi-stable if as $x \to \infty$, $y(x)$ goes to c on one side, but diverges on the other side.

Determining stability using sign of y'

- $y' = 0$ at equilibrium.
- $y' > 0$ implies $y(x)$ gets bigger
- $y' < 0$ implies $y(x)$ gets smaller

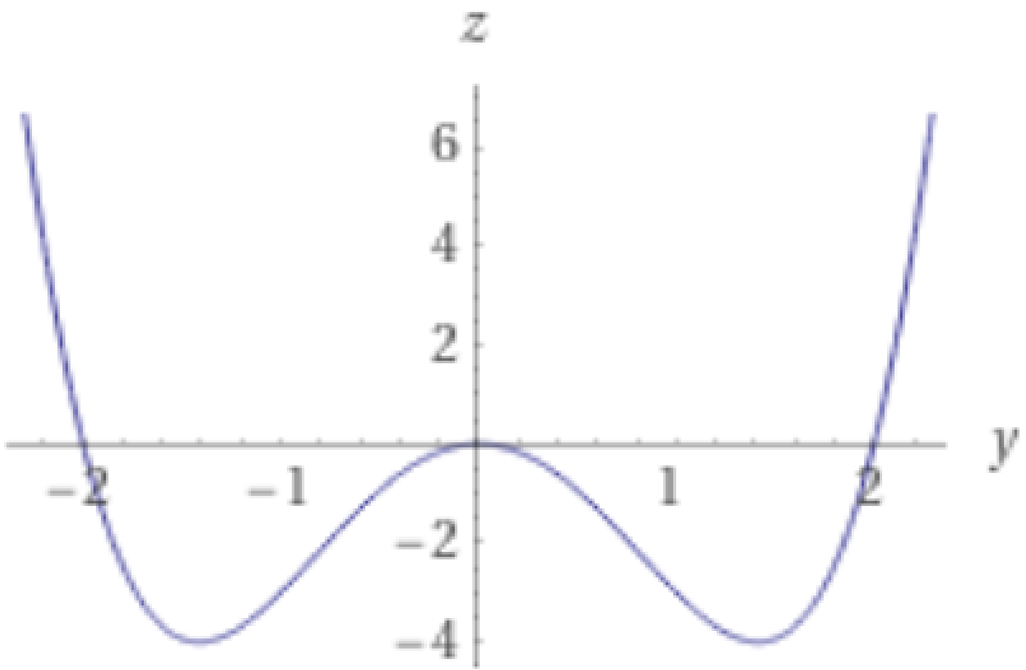
$y' = y^2 - y - 6$



Phase line

- Draw arrows along the y-axis depending on if $z = f(y) = y'$ is positive or negative.

$$y' = y^4 - 4y^2$$

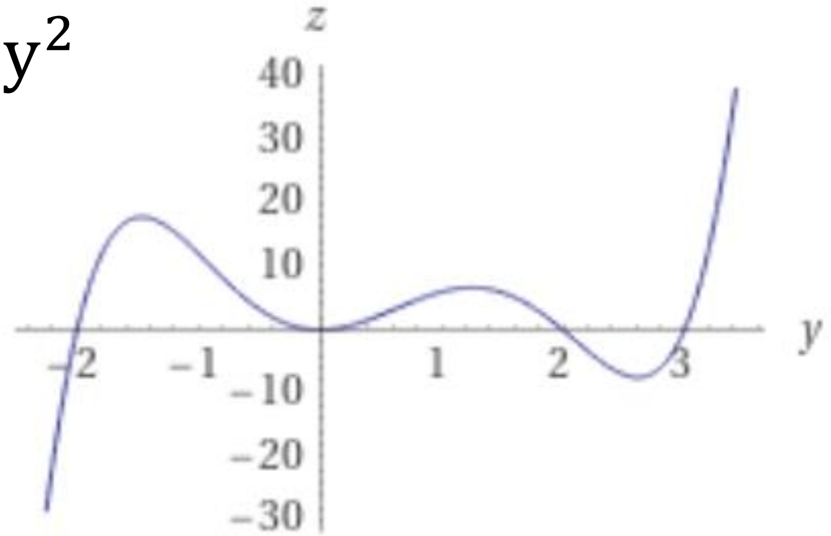


Phase line stability of $y' = f(y)$

- If $\mathrm{z} = f(y)$ crosses the y-axis at $f(c) = 0$ going upward, then c is unstable.
- If $\mathrm{z} = f(y)$ crosses the y-axis at $f(c) = 0$ going downward, then c is asymptotically stable.
- If $\mathrm{z} = f(y)$ touches the y-axis at $f(c) = 0$, but remains on the same side of the y-axis, then c is semi-stable.

Try it out

- $y' = y^5 - 3y^4 - 4y^3 + 12y^2$
- Classify the stability of the following equilibria:
- $y = -2$
- $y = 0$
- $y = 2$
- $y = 3$



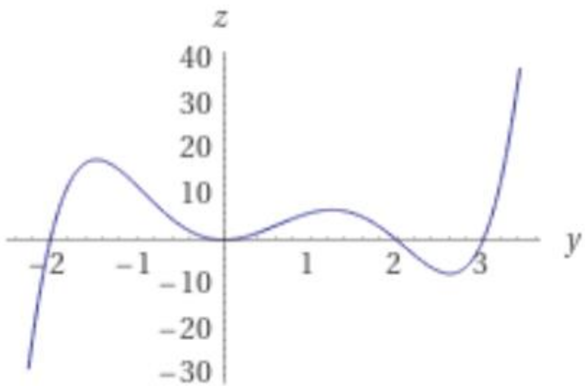
A: Asymptotically stable
B: Unstable
C: Semi-stable
D: ???
E: None of the above

Derivative test for stability

- Let $y' = f(y)$ have an equilibrium at $f(c) = 0$.
- If $\frac{df}{dy}(c) > 0$, then $y = c$ is unstable.
- If $\frac{df}{dy}(c) < 0$, then $y = c$ is asymptotically stable.
- If $\frac{df}{dy}(c) = 0$ and c is a local extremum (max or min) of $f(y)$, then $y = c$ is semi-stable

Example

- $y' = \mathrm{y}^5 - 3\mathrm{y}^4 - 4\mathrm{y}^3 + 12\mathrm{y}^2$ has equilibria -2, 0, 2, 3



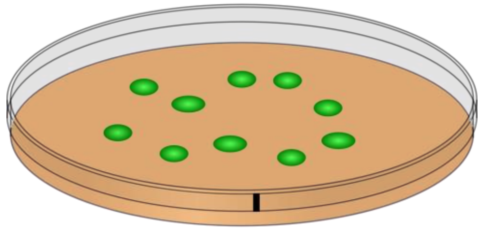
Try it out

- Fat crystallization: Let $y(x)$ be the proportion of crystallizable milk fat in a sample after x hours, satisfying
$$y' = 8(y^5 - y)$$
- If you start with half of the fat as crystallizable, how much fat is crystallizable as time goes to ∞?

https://commons.wikimedia.org/wiki/File:Glass_of_milk.jpg

A: All of the fat
B: Half of the fat
C: None of the fat
D: ???
E: None of the above

Logistic Growth Model



- Previously, we saw exponential growth $y' = ky$, $y(0) = y_0$, which had a solution $y(x) = y_0 e^{kx}$.
- In practice, this is unrealistic. For example, bacteria in a petri dish will initially grow almost exponentially, but then they'll use up all the available media.
- A better model is the logistic model, $y' = ky\left(1 - \frac{y}{L}\right)$, where the parameter L is the carrying capacity of the environment, and $k > 0$ is still the growth rate.

Stability of equilibria of logistic model

- $y' = ky\left(1 - \frac{y}{L}\right)$

Logistic model behavior

- $y' = ky\left(1 - \frac{y}{L}\right)$
- $y < 0$ is not physical, since we cannot have negative bacteria.
- When $0 < y < L$, $y' > 0$, so the number of bacteria increase, up to L.
- When $y > L$, $y' < 0$, so the number of bacteria decrease, down to L.
- On large time-scales, we therefore have L bacteria.