# Compartmental Models
# Lecture 8a: 2023-03-06

MAT A35 – Winter 2023 – UTSC

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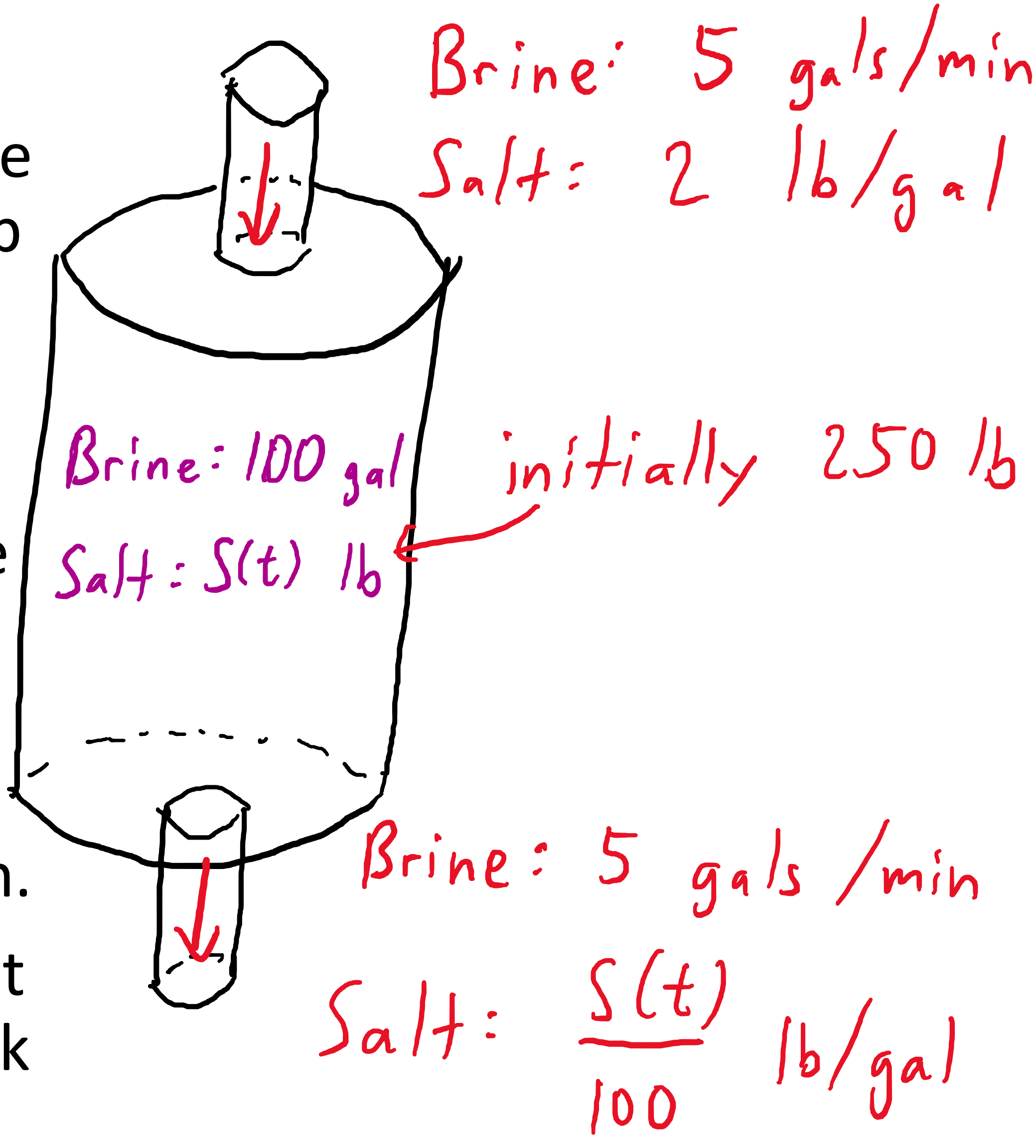
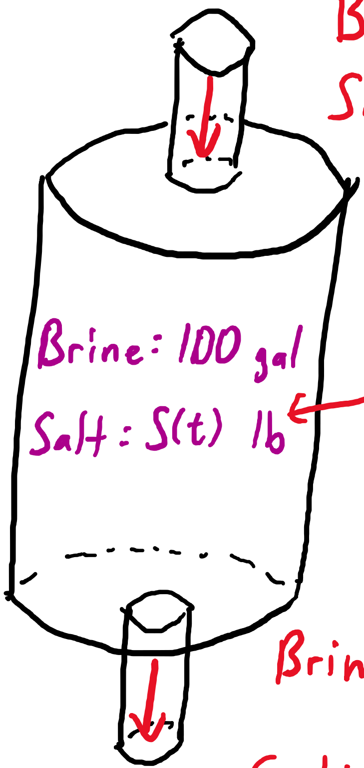
# Mixing problem (Bittinger, pg 559, ex. 7)

- A tank contains 100 gallons of brine whose concentration is 2.5 lb of salt / gallon.
- Brine containing 2 lb of salt / gallon runs into the tank at a rate of 5 gallons / min
- The brine in the tank runs out at the same rate of 5 gallons / min.
- How does the amount of salt $S(t)$ in the tank change over time?



# Rate of change in amount of salt




Brine: 5 gals/min
Salt: 2 lb/gal

10 lb/min of salt being added

initially 250 lb

$$\boxed{\frac{dS}{dt} = \dot{S} = 10 - \frac{S}{20}} \quad \text{lb/min}$$

Brine: 5 gals/min
Salt: $\frac{S(t)}{100}$ lb/gal

$\frac{S(t)}{20}$ lb/min of salt leaving the tank

# Solving for $\dot{S} = 10 - \frac{S}{20}$

$\dot{s} = \frac{dS}{dt}$

$$\dot{S} + \underbrace{\frac{1}{20}}_{p(x)} S = \underbrace{10}_{q(x)}$$

linear 1st order ODE

$$dS + dt\left[\frac{1}{20}\right]S = 10\,dt$$

$$I(t) = e^{\int \frac{1}{20}dt} = e^{\frac{t}{20}}$$

(integrating factor)

$$\int e^{\frac{t}{20}}\left[dS + dt\left[\frac{1}{20}S\right]\right] = \int 10e^{\frac{t}{20}}dt$$

$$\left(e^{-t/20}\right)\cdot\left[e^{t/20}\cdot S\right] = \left[200e^{\frac{t}{20}} + C\right]\cdot\left(e^{-t/20}\right)$$

$$S = 200 + Ce^{-\frac{t}{20}}$$

← general sol

# Initial value problem

- $S = 200 + Ce^{-\frac{t}{20}}$ and $S(0) = 250$

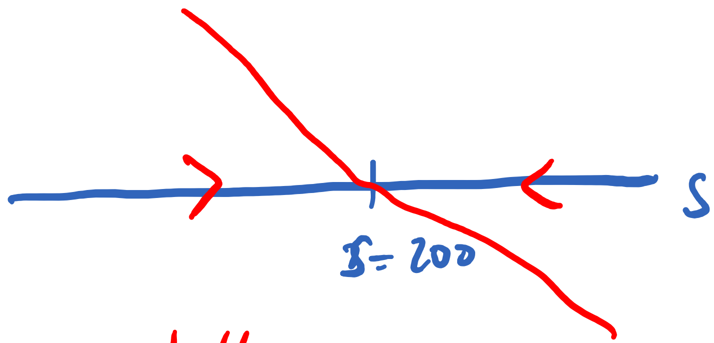
$S$ is in lbs
$t$ is in min
$\dot{S} = \frac{dS}{dt}$ is in lb/min

$\dot{S} = 10 - \frac{S}{20}$

$250 = S(0) = 200 + C$

$C = 50$

$S = 200 + 50e^{-t/20}$

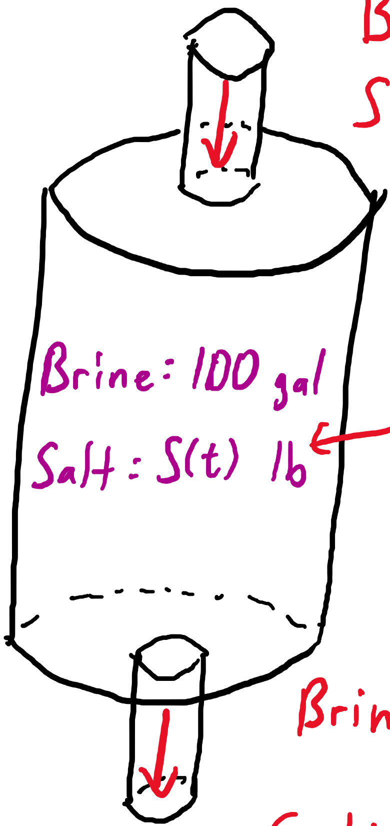



stable eq.

Amount of salt in tank on long time-scales?
A: 50 lb
B: 50 lb / min
C: 200 lb
D: 200 lb / min
E: None of the above

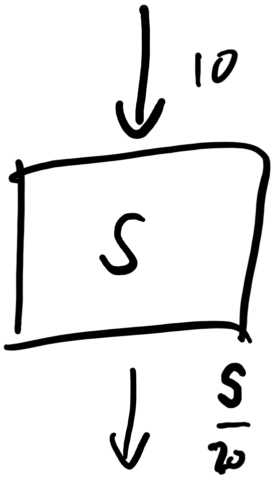
# Compartmental diagram: $S' = 10 - \frac{S}{10}$

Brine: 5 gals/min
Salt: 2 lb/gal




initially 250 lb

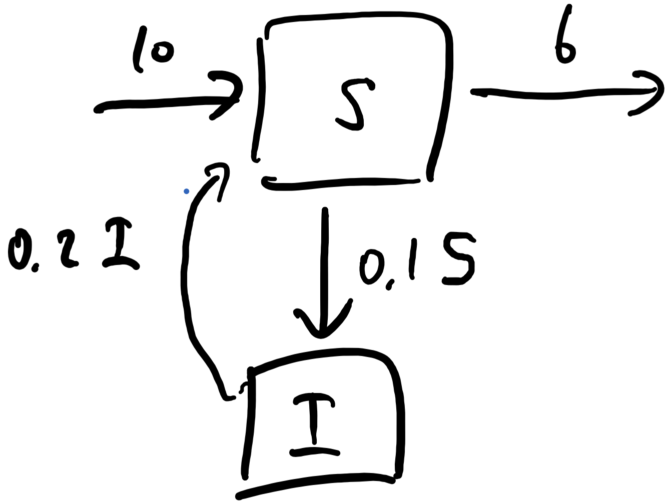
Brine: 5 gals/min
Salt: $\frac{S(t)}{100}$ lb/gal



# Compartmental models

- Boxes represent variables
- Arrows give rates of change.
  - An arrow pointing to a box increases that variable
  - An arrow pointing away decreases that variable

Ex.




$$S' = 10 - 6 - 0.1S + 0.2I$$

$$I' = 0.1S - 0.2I$$

# Application: SIR epidemic model

- Consider a simple epidemic model with three classes of people:
  - Susceptible individuals $S(t)$
  - Infected individuals $I(t)$
  - Removed individuals $R(t)$
- Assume that:
  - Infection rate is proportional to the number of infected individuals multiplied by the proportion of susceptible individuals in the population $N = S + I + R$. $\beta$
  - Recovery rate is proportional to the number of infected individuals $\gamma$
  - Recovered individuals are immune

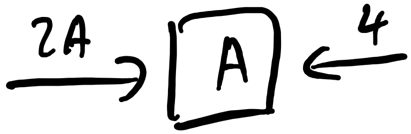
$$\boxed{S} \xrightarrow{\beta \cdot \frac{SI}{N}} \boxed{I} \xrightarrow{\gamma I} \boxed{R}$$

$$\begin{cases} \dot{S} = -\beta \cdot \dfrac{SI}{N} \\ \dot{I} = \dfrac{\beta SI}{N} - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

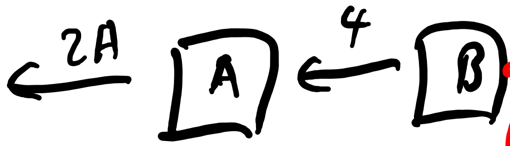
# Try it out: $A' + 2A = 4$

$A' = -2A + 4$

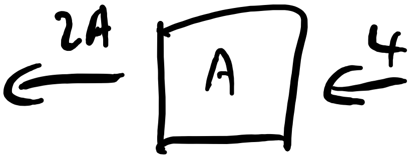
- Does the following compartmentalized model correspond to the equation above?



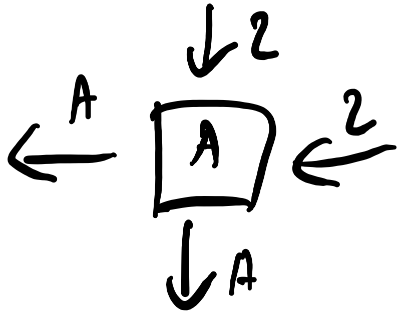

NO




$$\begin{cases} A' = -2A + 4 \\ B' = -4 \end{cases}$$




YES



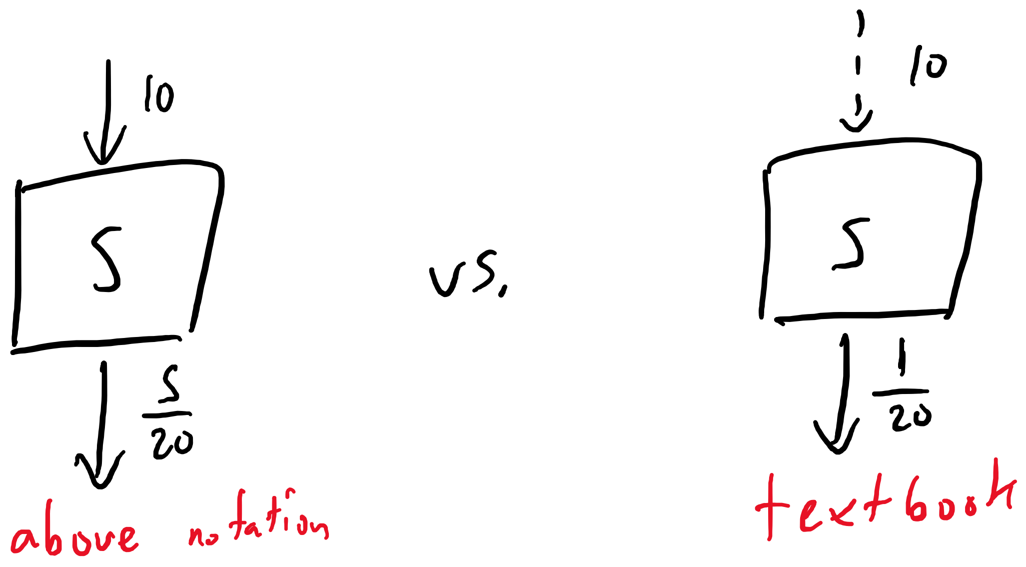

YES

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

# Notational aside

- Bittinger, et al textbook introduces two different types of arrows
  - Relative rates: solid arrows are proportional to the variables
  - Constant/time-dependent rates: dotted arrows are either constant or depend only on time

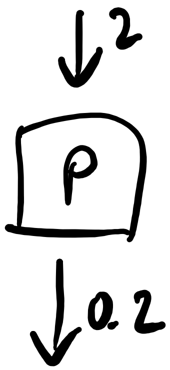
Ex.




- MATA35 will use the left-hand notation of using solid lines everywhere and including all relevant variables unless explicitly specified otherwise.

# Try it out

- Find the general solution for the following one-compartment model:




$\dot{P} = 2 - 0.2 = 1.8$

$P = 1.8t + C$


$\dot{P} = 2 - 0.2P$

$\dot{P} + 0.2P = 2$

$I(t) = e^{0.2t}$

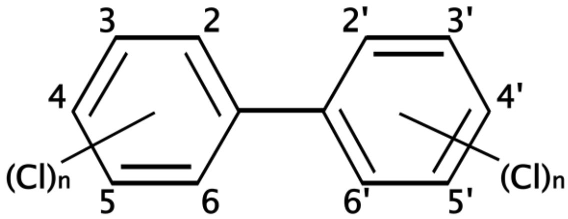
$Pe^{0.2t} = 10e^{0.2t} + C$

$P = 10 + Ce^{-0.2t}$

A: $P(t) = 1.8t + C$
B: $P(t) = 2.2t + C$
C: $P(t) = 10 + Ce^{-0.2t}$
D: ???
E: None of the above

# Lake pollution (Bittinger, pg 557, ex. 5)

- A unpolluted mussel is placed into lake water polluted by polychlorinated biphenyls (PCBs).
- Let $Q(t)$ be the concentration of PCB in the mussel in micrograms per gram of tissue.
- The mussel absorbs PCBs at 12 micrograms per gram of tissue each day.
- The mussel eliminates PCBs at a rate of $0.18Q$ micrograms per gram of tissue each day.




$\downarrow 12$

$\boxed{Q}$

$\downarrow 0.18Q$

$$\frac{dQ}{dt} = \dot{Q} = 12 - 0.18Q$$

$$\dot{Q} + 0.18Q = 12$$

$$I(t) = e^{0.18t} \leftarrow \text{Integrating Factor}$$

$$Qe^{0.18t} = \int 12e^{0.18t}\,dt = \frac{200}{3}e^{0.18t} + C$$

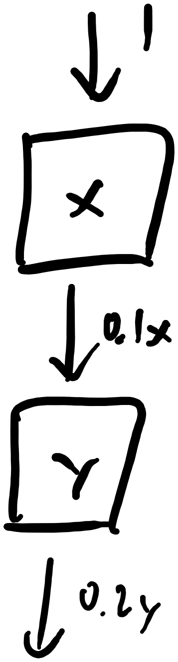
$$Q(t) = \frac{200}{3} + Ce^{-0.18t}$$

- $Q(0) = 0$

$$C = -\frac{200}{3}$$

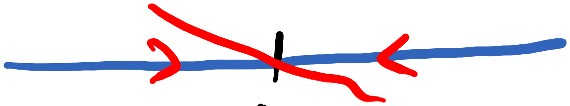
$$Q(t) = \frac{200}{3} - \frac{200}{3}e^{-0.18t}$$

# Simple multi-compartmental models



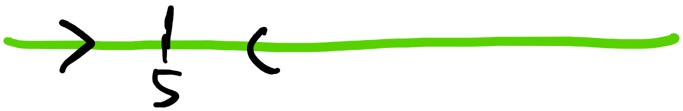

$$\begin{cases} \dot{x} = 1 - 0.1x \\ \dot{y} = 0.1x - 0.2y \end{cases}$$

Asymp. stable eq for x



$\dot{x} = 0 = 1 - 0.1x \Rightarrow x = 10$

Asymp stable eq for y




At time $\infty$, $\dot{y} = 0.1x - 0.2y = 1 - 0.2y = 0$

$\Rightarrow y = 5$

At time $\infty$,

$x = 10$

$y = 5$

# Solving explicitly


$$\begin{cases} \dot{x} = 1 - 0.1x \\ \dot{y} = 0.1x - 0.2y \end{cases}$$

$$\dot{x} + 0.1x = 1$$

$$I(t) = e^{0.1t}$$

$$x e^{0.1t} = \int e^{0.1t}\, dt$$

$$x e^{0.1t} = 10 e^{0.1t} + C_1$$

$$x = \underline{10} + C_1 e^{-0.1t}$$

$$\dot{y} = 0.1\left(10 + C_1 e^{-0.1t}\right) - 0.2y$$

$$\dot{y} + 0.2y = 1 + 0.1 C_1 e^{-0.1t}$$

$$y e^{0.2t} = \int \left[e^{0.2t} + 0.1 C_1 e^{0.1t}\right] dt$$

$$y e^{0.2t} = 5 e^{0.2t} + C_1 e^{0.1t} + C_2$$

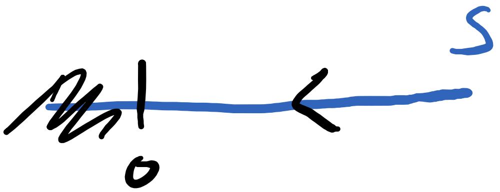
$$y = \underline{5} + C_1 e^{-0.1t} + C_2 e^{-0.2t}$$

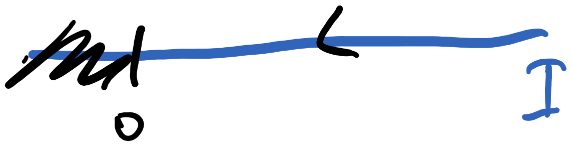
# Try it out

- What is the behavior of the basic SIR epidemic model with no reinfection at time infinity, if $S(0) > 0, I(0) > 0$, and $R(0) > 0$?

$$S \xrightarrow{\frac{\beta S I}{N}} I \xrightarrow{\gamma I} R$$

$$R = N - S - I$$




$$\lim_{t \to \infty} R(t) = N - 0 - 0 = N$$

$$\dot{S} = \frac{-\beta S I}{N}$$

$$\dot{I} = \underbrace{\frac{\beta S I}{N}}_{0 \text{ at } t = \infty} - \gamma I$$

$$\lim_{t \to \infty} S(t) = 0$$

$$\lim_{t \to \infty} I(t) = 0$$

A: Everyone is susceptible
B: Everyone is infected
C: Everyone is removed
D: Some mix of the above
E: None of the above

# Concluding remarks

- Compartmental models are a graphical way of representing the rate of change of variables.
- Linear one-compartment models we can solve using the techniques for linear first-order ODEs
  - Explicitly solving using integrating factors
  - Stability analysis using phase lines
- Multi-compartment models that can be broken down into a series of one-compartment models are also solvable for the same reason.
- More complicated multi-compartment models will require knowing how to solve systems of ODEs. (Future lecture)