Substitution Method
Lecture 8b: 2023-03-06

MAT A35 – Winter 2023 – UTSC

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Recall substitution method for integrals

- Step 1: Guess an appropriate u
- Step 2: Compute du, dx, and x
- Step 3: Substitute in to get rid of all the x's
- Step 4: Integrate as a function of u
- Step 5: Convert back to x's

$y = xe^{x^2}$

$\frac{dy}{dx} = xe^{x^2}$

$dy = xe^{x^2}dx$

$y = \int xe^{x^2}dx$

1. Let $u = x^2$

2. $du = 2x\,dx$

3. $y = \int xe^{x^2}dx = \frac{1}{2}\int e^u du$

4. $y = \frac{1}{2}e^u + C$

5. $y = \frac{1}{2}e^{x^2} + C$

Substitution method for ODEs

- Goal: convert ODE to a form we know how to solve.
- Strategy: Guess an appropriate $u = f(x, y)$ to simplify the ODE.
- Caveat: Sometimes need multiple substitutions.

Ex. $$2ye^{x/y}\,dx + \left(y - 2xe^{x/y}\right)dy = 0$$

Let $u = \frac{x}{y}$ $\Rightarrow$ $x = uy,\quad dx = u\,dy + y\,du$

$$\Rightarrow 2ye^{u}\left[u\,dy + y\,du\right] + \left(y - 2uye^{u}\right)dy = 0$$

$$dy\left[\underline{2uye^{u}}\ \underline{-2uye^{u}} + y\right] + du\left[2y^{2}e^{u}\right] = 0$$

$$y\,dy + du\left[2y^{2}e^{u}\right] = 0$$

$$\int \frac{1}{y}\,dy + \int 2e^{u}\,du = 0$$

$$\ln|y| + 2e^{u} = C$$

$$\ln|y| + 2e^{x/y} = C$$

A much more complicated example

- $(2x + y + 1)dx + (x - y - 4)dy = 0$
- Multistep derivation:
  - Substitute $\begin{cases} u = 2x + y + 1 \\ v = x - y - 4 \end{cases}$
  - Substitute $w = \frac{u}{v}$
  - We now have a separable equation.
  - Solve via integrating both pieces.
  - Undo all of the substitutions

$$\text{Substitute}\begin{cases} u = 2x + y + 1 \\ v = x - y - 4 \end{cases}$$

- $(2x + y + 1)dx + (x - y - 4)dy = 0$

$$\left.\begin{aligned} du &= 2dx + dy \\ dv &= dx - dy \end{aligned}\right\} \qquad \begin{aligned} du + dv &= 3dx \\ dx &= \frac{du + dv}{3} \end{aligned} \qquad \begin{aligned} dy &= dx - dv \\ dy &= \frac{du - 2dv}{3} \end{aligned}$$

$$u\left(\frac{du + dv}{3}\right) + v\left(\frac{du - 2dv}{3}\right) = 0$$

$$u(du + dv) + v(du - 2dv) = 0$$

$$(u + v)du + (u - 2v)dv = 0$$

Substitute $w = \frac{u}{v}$

$$(u+v)\,du + (u-2v)\,dv = 0$$

Let $w = \frac{u}{v}$ $\qquad u = wv \qquad du = v\,dw + w\,dv$

$$v(w+1)(v\,dw + w\,dv) + v(w-2)\,dv = 0$$

$$dw\,[v(w+1)] + dv\,[w^2 + w + w - 2] = 0$$

$$dw\,[v(w+1)] + dv\,[w^2 + 2w - 2] = 0$$

Separation of variables

$$dw\left[v(w+1)\right] + dv\left[w^2+2w-2\right] = 0$$

$$dw\left[\frac{w+1}{w^2+2w-2}\right] + dv\left[\frac{1}{v}\right] = 0$$

Let $z = w^2+2w-2$

$dz = (2w+2)dw$

$$\int \frac{1}{v}\,dv = -\int \frac{w+1}{w^2+2w-2}\,dw$$

$$\ln|v| = -\int \frac{\frac{1}{2}dz}{z} = -\frac{1}{2}\ln|z| + C$$

$$|v| = \frac{C}{\sqrt{|z|}}$$

$$|v| = \frac{C}{\sqrt{|w^2+w-2|}} \Rightarrow |v|\sqrt{|w^2+2w-2|} = C$$

Substituting everything back in

Substitute $w = \frac{u}{v}$, and Substitute $\begin{cases} u = 2x + y + 1 \\ v = x - y - 4 \end{cases}$

$$|v| \sqrt{|w^2 + 2w - 2|} = C$$

$$|v| \sqrt{\left| \frac{u^2}{v^2} + 2\frac{u}{v} - 2 \right|} = C$$

$$|x - y - 4| \sqrt{\left| \frac{(2x+y+1)^2}{(x-y-4)^2} + 2 \cdot \frac{(2x+y+1)}{(x-y-4)} - 2 \right|} = C$$

Try it out

$$2u - 13$$

$$2x + 4y + 10 - 13$$

- $(x + 2y + 5)dx + (2x + 4y - 3)dy = 0.$
  Let $u = x + 2y + 5$

$x = u - 2y - 5$

Let $du = dx + 2dy$

$dx = du - 2dy$

$$u(du - 2dy) + [2(u - 2y - 5) + 4y - 3]dy = 0$$

$$u(du - 2dy) + (2u - 13)dy = 0$$

$$u\,du - 13\,dy = 0$$

$$\frac{1}{2}u^2 - 13y = 0$$

$$\frac{1}{2}(x + 2y + 5)^2 - 13y = C$$

A: $(x + 2y + 5)y^2 = C$
B: $\frac{1}{2}(x + 2y + 5)^2 - 13y = C$
C: $(x + 2y + 5) - 13\mathrm{y}^2 = \mathrm{C}$
D: All of the above
E: None of the above

Common Substitution Guesses

- $P(x,y)dx + Q(x,y)dy = 0$, where $P(tx,ty) = t^n P(x,y)$ and $Q(tx,ty) = t^n Q(x,y)$ for some integer n.
  - Let $u = x/y$. Will get separable ODE.
  - Special case: $(a_1x + b_1y)dx + (a_2x + b_2y)dy = 0$
- $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$
  - If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are intersecting lines, then let both $\begin{cases} u = a_1x + b_1y + c_1 \\ v = a_2x + b_2y + c_2 \end{cases}$. Will get case above.
  - If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel lines, then let $u = a_1x + b_1y + c_1$ or let $u = a_2x + b_2y + c_2$. Will get separable ODE.

$a_1x + b_1y + c_1 = 0$

$y = \frac{-a_1x}{b_1} - \frac{c_1}{b_1}$

- Bernoulli ODE: $\frac{dy}{dx} + P(x)y = Q(x)y^n$
  - Multiply by $(1-n)y^{-n}$. Then Let $u = y^{1-n}$.
  - Will get linear first-order ODE.

Guess the substitution

- $(x+y-1)dx + (x+y+1)dy = 0$

$y = -x + 1$ $\qquad$ $y = -x - 1$

A: $u = x + y - 1$
B: $v = x + y + 1$
C: Both A & B
D: One of A or B
E: $u = x/y$

- $(x+y-1)dx + (2x-y)dy = 0$

1) Substitute both A + B

2) $z = \frac{u}{v}$

step 2

Step 1

A: $u = x + y - 1$
B: $v = 2x - y$
C: Both A & B
D: One of A or B
E: $u = x/y$

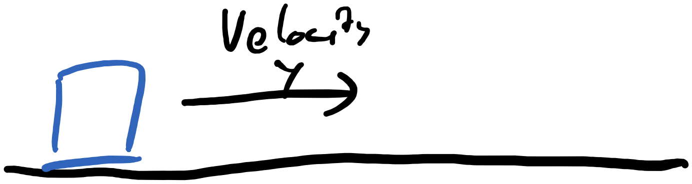
- $\exp\left(\frac{x}{y}\right) dx + \tan\left(\frac{x}{y}\right) dy = 0$

Let $u = \frac{x}{y}$

A: $u = \exp\left(\frac{x}{y}\right)$
B: $v = \tan\left(\frac{x}{y}\right)$
C: Both A & B
D: One of A or B
E: $u = x/y$

Application: friction and drag

Time x



Drag and friction

$$y' = -y + 0.01y^3$$

$$y' + y = -0.01y^3$$

Bernoulli

Caveats

- Like Integrating Factors, guessing the right substitution is hard outside of a few known special cases, like the ones in the previous slide.
- On Quiz 4, I will give you the appropriate substitution or provide you with the information on the common substitutions slide.
- Sometimes, may even just ask you to guess an appropriate series of substitutions to convert the ODE to a different form. (e.g. to a separable ODE)