Substitution Method
Lecture 8b: 2023-03-06

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

Recall substitution method for integrals

- Step 1: Guess an appropriate u
- Step 2: Compute du, dx, and x
- Step 3: Substitute in to get rid of all the x's
- Step 4: Integrate as a function of u
- Step 5: Convert back to x's

Substitution method for ODEs

- Goal: convert ODE to a form we know how to solve.
- Strategy: Guess an appropriate $u = f(x, y)$ to simplify the ODE.
- Caveat: Sometimes need multiple substitutions.

A much more complicated example

- $(2x + y + 1)dx + (x - y - 4)dy = 0$
- Multistep derivation:
  - Substitute $\begin{cases} u = 2x + y + 1 \\ v = x - y - 4 \end{cases}$
  - Substitute $w = \frac{u}{v}$
  - We now have a separable equation.
  - Solve via integrating both pieces.
  - Undo all of the substitutions

Substitute $\begin{cases} u = 2x + y + 1 \\ v = x - y - 4 \end{cases}$

- $(2x + y + 1)dx + (x - y - 4)dy = 0$

Substitute $w = \dfrac{u}{v}$

Separation of variables

Substituting everything back in

Substitute $w = \frac{u}{v}$, and Substitute $\begin{cases} u = 2x + y + 1 \\ v = x - y - 4 \end{cases}$

Try it out

- $(x + 2y + 5)dx + (2x + 4y - 3)dy = 0.$
  Let $u = x + 2y + 5$

A: $(x + 2y + 5)y^2 = C$
B: $\frac{1}{2}(x + 2y + 5)^2 - 13y = C$
C: $(x + 2y + 5) - 13\mathrm{y}^2 = \mathrm{C}$
D: All of the above
E: None of the above

Common Substitution Guesses

- $P(x, y)dx + Q(x, y)dy = 0$, where $P(tx, ty) = t^n P(x, y)$ and $Q(tx, ty) = t^n Q(x, y)$ for some integer n.
  - Let $u = x/y$. Will get separable ODE.
  - Special case: $(a_1 x + b_1 y)dx + (a_2 x + b_2 y)dy = 0$
- $(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$
  - If $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are intersecting lines, then let both $\begin{cases} u = a_1 x + b_1 y + c_1 \\ v = a_2 x + b_2 y + c_2 \end{cases}$. Will get case above.
  - If $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are parallel lines, then let $u = a_1 x + b_1 y + c_1$ or let $u = a_2 x + b_2 y + c_2$. Will get separable ODE.
- Bernoulli ODE: $\frac{dy}{dx} + P(x)y = Q(x)y^n$
  - Multiply by $(1 - n)y^{-n}$. Then Let $u = y^{1-n}$.
  - Will get linear first-order ODE.

$a_1 x + b_1 y + c_1 = 0$

$y = \frac{-a_1 x}{b_1} - \frac{c_1}{b_1}$

Guess the substitution

- $(x+y-1)dx+(x+y+1)dy=0$

A: $u=x+y-1$
B: $v=x+y+1$
C: Both A & B
D: One of A or B
E: $u=x/y$

- $(x+y-1)dx+(2x-y)dy=0$

A: $u=x+y-1$
B: $v=2x-y$
C: Both A & B
D: One of A or B
E: $u=x/y$

- $\exp\left(\frac{x}{y}\right)dx+\tan\left(\frac{x}{y}\right)dy=0$

A: $u=\exp\left(\frac{x}{y}\right)$
B: $v=\tan\left(\frac{x}{y}\right)$
C: Both A & B
D: One of A or B
E: $u=x/y$

Application: friction and drag

Caveats

- Like Integrating Factors, guessing the right substitution is hard outside of a few known special cases, like the ones in the previous slide.

- On Quiz 4, I will give you the appropriate substitution or provide you with the information on the common substitutions slide.
- Sometimes, may even just ask you to guess an appropriate series of substitutions to convert the ODE to a different form. (e.g. to a separable ODE)