Numerical solutions: Euler’s Method and Runge-Kutta
Lecture 8c: 2023-03-09

MAT A35 – Winter 2023 – UTSC

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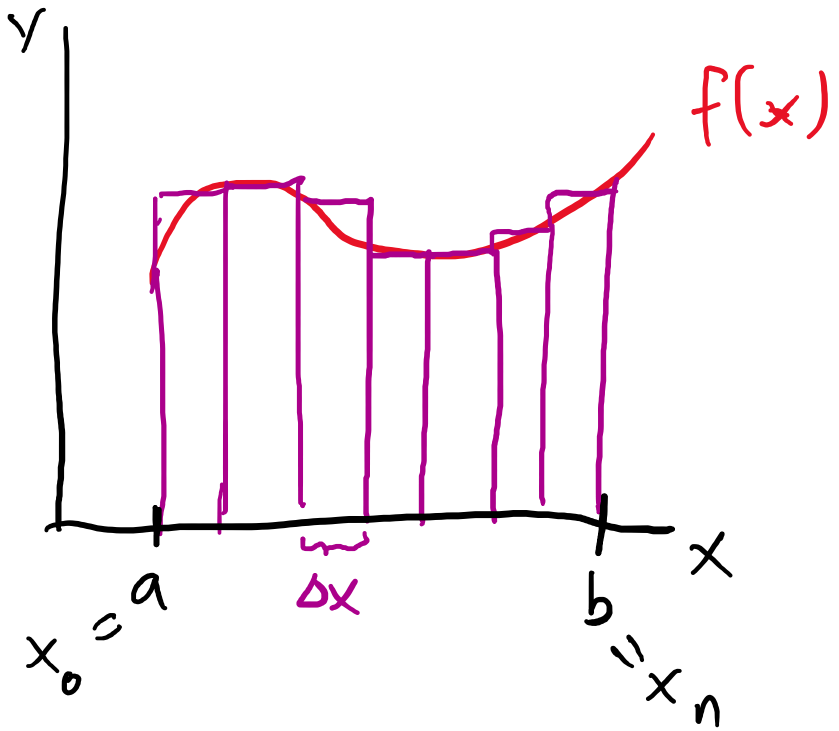
Recall: Riemann Sums

- For any integral problem, we can approximate it with lots of little rectangles. The approximation gets better the more rectangles we have.



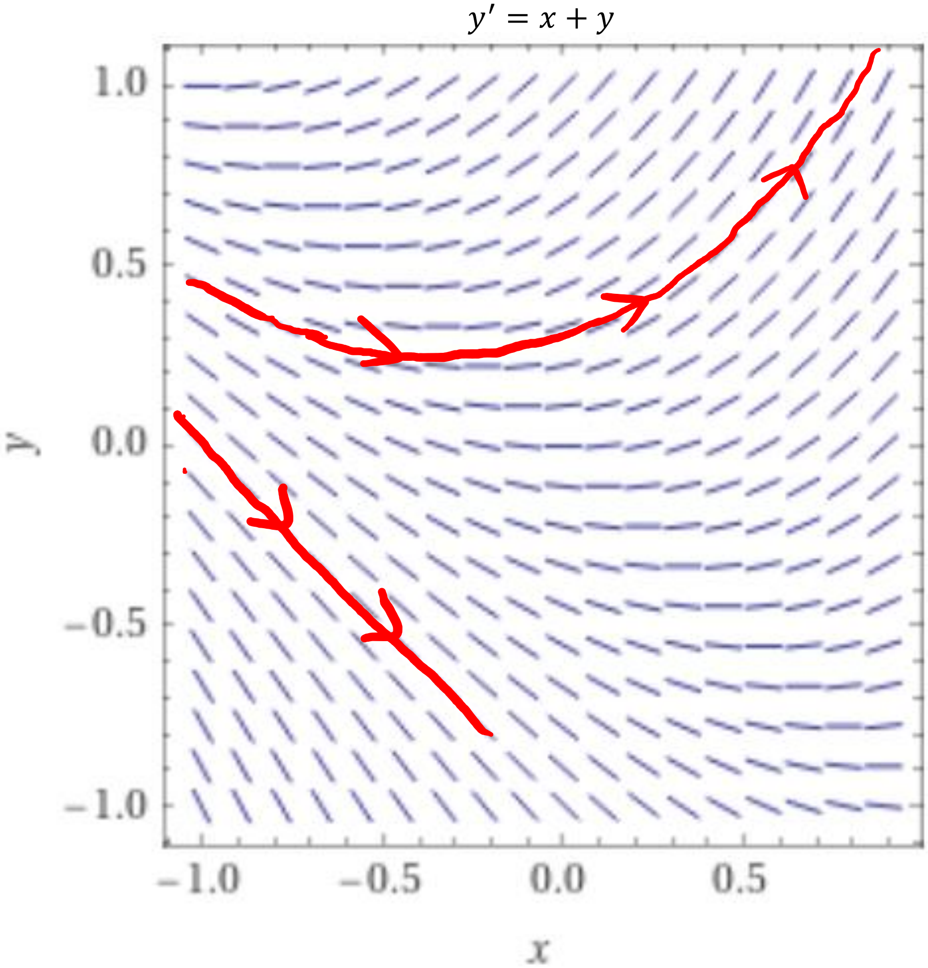
$$\int_a^b f(x)\,dx$$

$\approx$ signed area in rectangles

$$= \sum_{i=1}^{n} \Delta x\, f(x_i)$$

Recall: direction fields

- Direction fields tell you what direction a solution to the ODE goes.
- We can approximate a solution to the ODE by starting somewhere and following the direction field.



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Euler's Method

Initial value problem

- Suppose we have an IVP
$\boxed{y' = f(x, y)}, \qquad y(x_0) = y_0$
- Choose a *step-size* Δx.
- Then $x_{i+1} = x_i + \Delta x$
- Let $y_{i+1} = y_i + \underbrace{f(x_n, y_n)}\Delta x$.
- Then $y_n \approx y(x_n)$.

$y_{i+1} - y_i = f(x_n, y_n)\Delta x$

$\Delta y_i = f(x_n, y_n)\Delta x$

$\approx \quad dy = f(x, y)\, dx$

$y' = x + y \qquad y(-1) = 0.5$

$\Delta x = 0.5$

$-1 + 0.5 = -0.5$

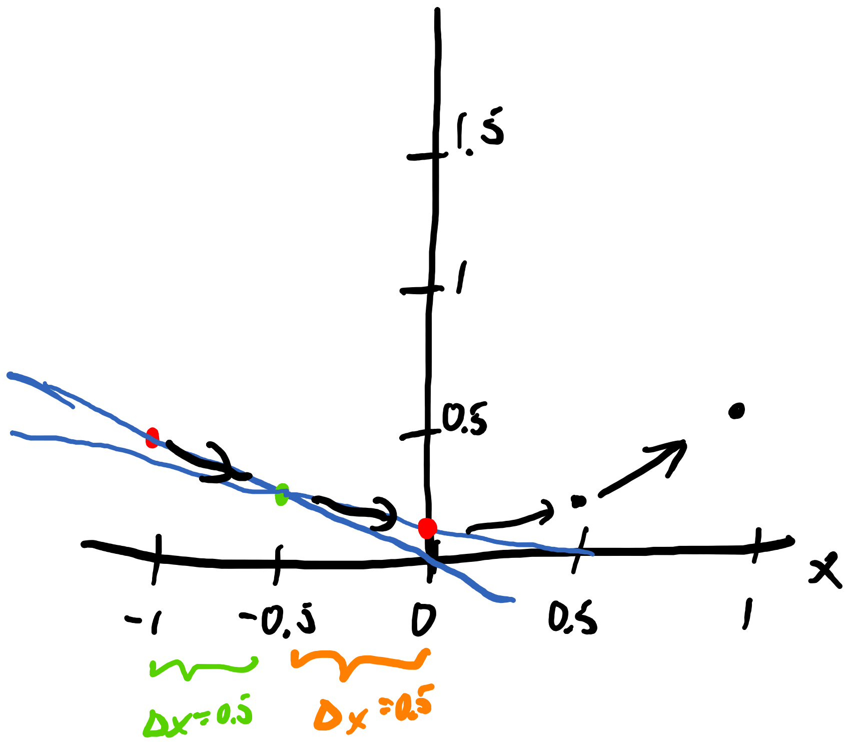
$x_0 = -1 \qquad y_0 = 0.5$

$x_1 = -0.5 \qquad y_1 = 0.5 + (-0.5)(0.5) = 0.25$

$x_2 = 0 \qquad y_2 = 0.25 + (-0.25)(0.5) = 0.125$

$x_3 = 0.5 \qquad y_3 = 0.125 + (0.125) \cdot 0.5 = 0.1875$

$x_4 = 1 \qquad y_4 = 0.1875 + (0.6875) \cdot 0.5 \approx 0.53$



Try it out

$\frac{dy}{dx}$ slope

$\frac{dy}{dx} = y$

$\int \frac{dy}{y} = \int dx$

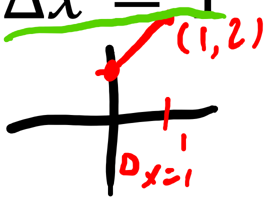
$\ln|y| = x + C$

$y = Ce^x$

$y = e^x \to y(1) = e$

≈ 2.718

- Consider $y' = y$, where $y(0) = 1$. Estimate $y(1)$ using Euler's method with the following step sizes
- $\Delta x = 1$



$x_0 = 0$, $y_0 = 1$ slope at $(0,1) = 1$

$x_1 = 0 + \Delta x = 1$ $y_1 = 1 + 1 \cdot 1 = 2$

- $\Delta x = \frac{1}{2}$

$x_0 = 0$ $y_0 = 1$ slope at $(0,1) = 1$

$x_1 = \frac{1}{2}$ $y_1 = 1 + 1 \cdot 0.5 = 1.5$ slope at $(\frac{1}{2}, 1.5) = 1.5$

$x_2 = 1$ $y_2 = 1.5 + 1.5 \cdot 0.5 = 2.25$

- $\Delta x = \frac{1}{3}$

$x_0 = 0$ $y_0 = 1$

$x_1 = \frac{1}{3}$ $y_1 = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3} \approx 1.\overline{333}$

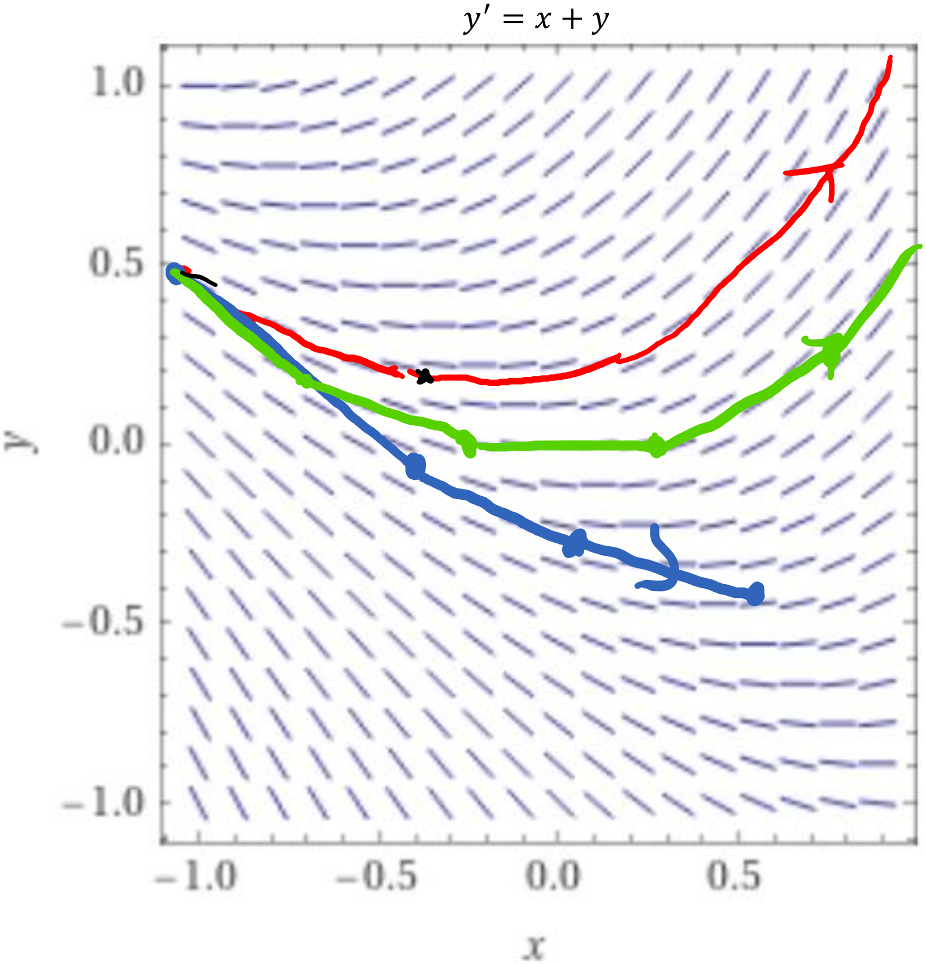
$x_2 = \frac{2}{3}$ $y_2 = \frac{4}{3} + \frac{4}{3} \cdot \frac{1}{3} = \left(\frac{4}{3}\right)^2 \approx 1.\overline{77}$

$x_3 = 1$ $y_3 = \left(\frac{4}{3}\right)^3 \approx 2.\overline{370}$

A: 2.718
B: 2.370
C: 2.250
D: 2.000
E: None of the above

Errors in Euler's method approximations

- We only use the slope at starting point of the integral, and the errors can accumulate.
- The smaller the step size, the more accurate the approximation, but also requires more computation time.



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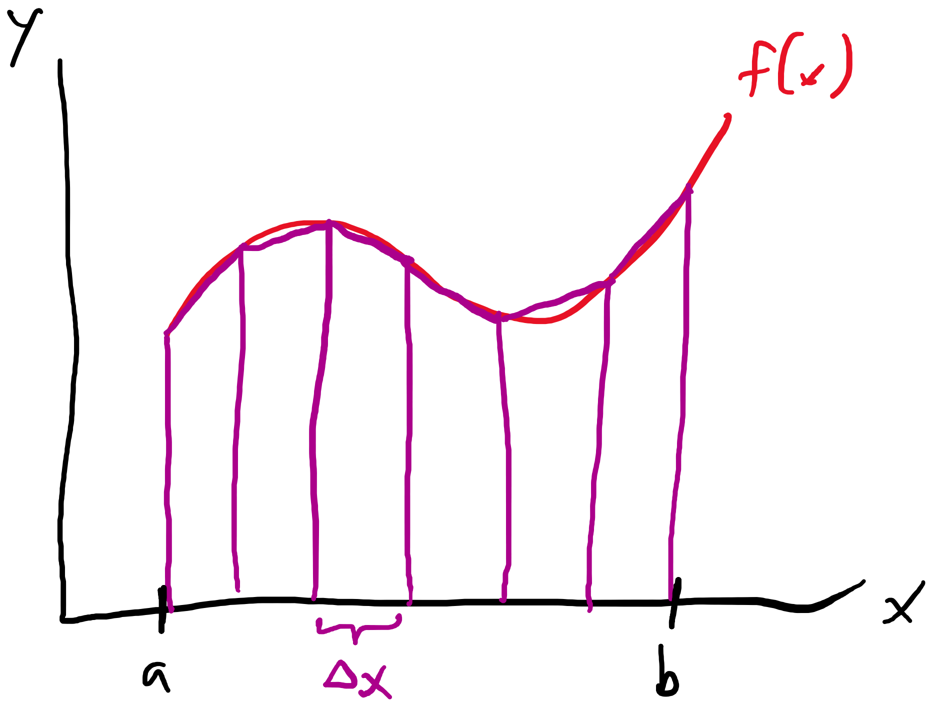
Recall: Trapezoid rule

- We can reduce the error of an integral by using both endpoints of an interval.



$$\int f(x)\,dx$$

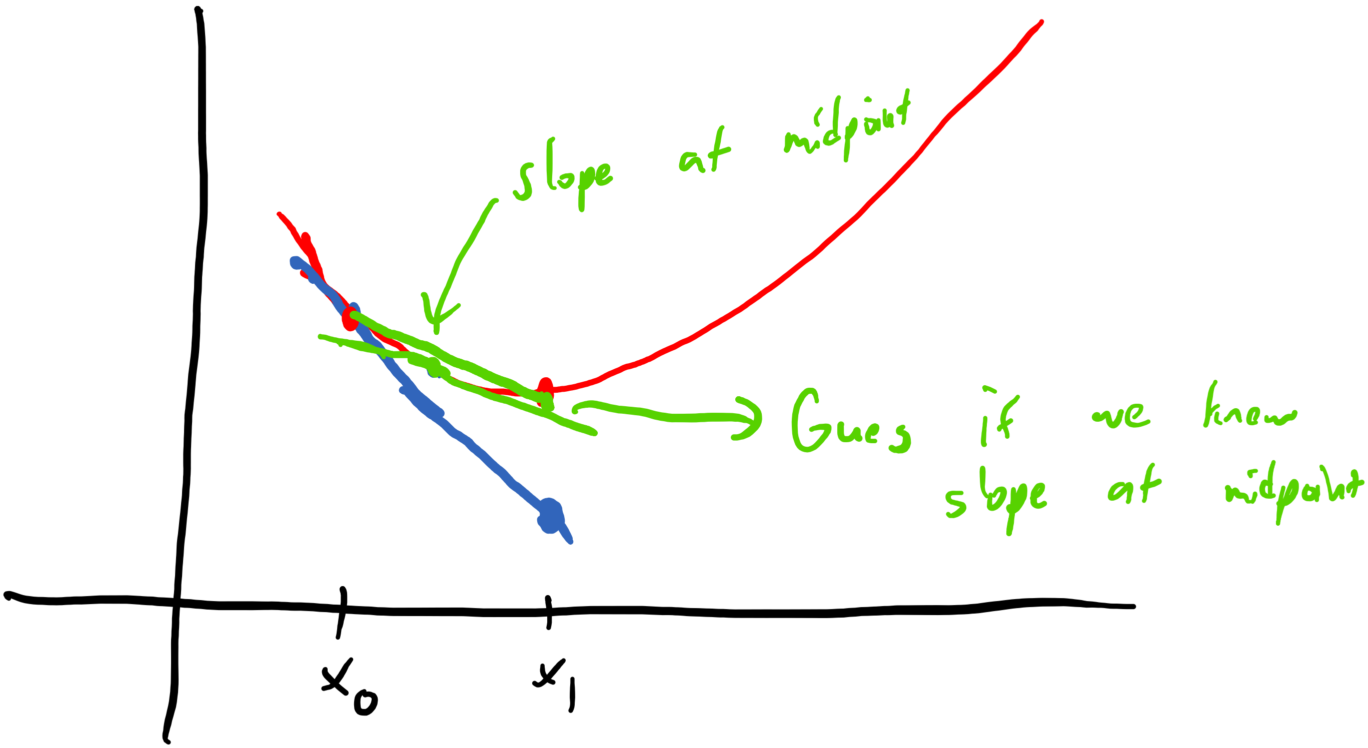
$\approx$ area in trapezoids

$$= \sum_{i=1}^{n} \frac{1}{2} \Delta x \left(f(x_i) + f(x_{i-1}) \right)$$

Runge-Kutta Family of Methods

- Euler's method is considered 1st-order Runge-Kutta
- Higher-order Runge-Kutta methods use multiple points to derive a better slope.

Intuition



Problem with intuition

- What's the biggest problem with the intuition on the previous slide?
  - A: We don't know where the midpoint is (in terms of (x, y) coordinates).
  - B: We know where the midpoint is, but cannot compute the slope there.
  - C: We know where the midpoint is, but its slope is not always a good estimate of the true slope.
  - D: Computing the midpoint takes a lot of computation.
  - E: None of the above

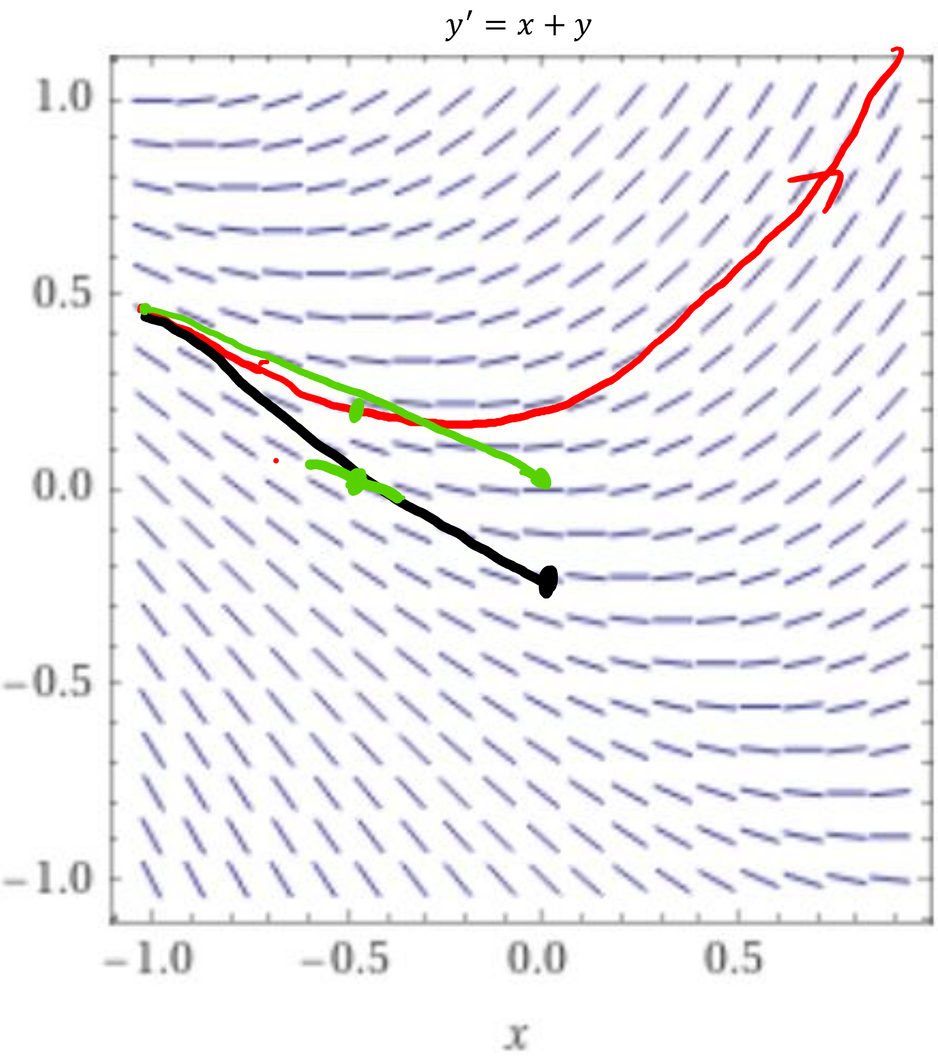
Secondary problem

Runge-Kutta – naïve 2nd order midpoint

Intuition :

slope at start

slope at guessed midpoint using slope at start



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Runge-Kutta – naïve 2nd order midpoint

- Suppose we have an IVP
$$y' = f(x, y), \qquad y(x_0) = y_0$$
- Choose a *step-size* Δx. Then $x_{i+1} = x_i + \Delta x$.
- Let $k_1 = f(x_i, y_i)$. ← slope at start
- Let $k_2 = f\left(\underbrace{x_i + \frac{\Delta x}{2}, y_i + \frac{k_1 \Delta x}{2}}_{\text{guessed midpoint based on } k_1 \text{ slope}}\right)$ } slope at guessed midpoint
- Let $\underbrace{y_{i+1} = y_i + k_2 \Delta x}_{\text{estimate using slope at guessed midpoint}}$.

Classic Runge-Kutta – 4th order

- Suppose we have an IVP
$$y' = f(x, y), \qquad y(x_0) = y_0$$
- Choose a *step-size* Δx. Then $x_{i+1} = x_i + \Delta x$.
- Let $k_1 = f(x_i, y_i)$. ← slope at start
- Let $k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1 \Delta x}{2}\right)$ ← slope at guessed mid pt based on k_1
- Let $k_3 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2 \Delta x}{2}\right)$ ← slope at guessed mid pt based on k_2
- Let $k_4 = f(x_i + \Delta x, y_i + \mathrm{k}_3 \Delta x)$ ← slope at guessed endpt based on k_3

- Let $y_{i+1} = y_i + \underbrace{\frac{1}{6}\Delta x(k_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4)}_{\text{weighted average of all 4 slopes}}$

Concluding remarks

- Like integrals, solving ODEs explicitly is often hard, and sometimes we don't have closed-form solutions.
- Like integrals, solving ODEs numerically is actually much easier, since we can approximate by taking lots of tiny Δx steps.
- Euler's method is similar to Riemann rectangular sums.
- Runge-Kutta (2nd order) is similar to Trapezoid rule.
- Runge-Kutta (classic, 4th order) is similar to Simpson's rule of thirds.
- In practice, we often solve complicated ODEs using these and other approximations.