Complex numbers and rotations Lecture 9a: 2023-03-13

MAT A35 – Winter 2023 – UTSC

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What were you doing at 2:30 am on Sunday?

A. Sleeping

B. Not existing

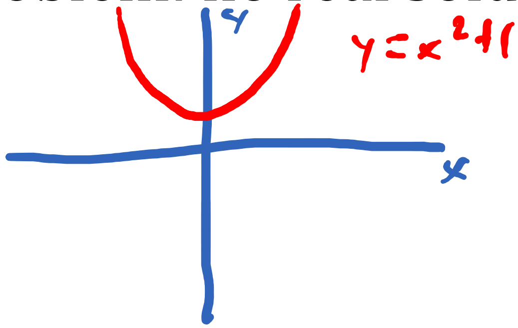
C. Lamenting the loss of an hour

D. Celebrating Daylight Savings

E. Other

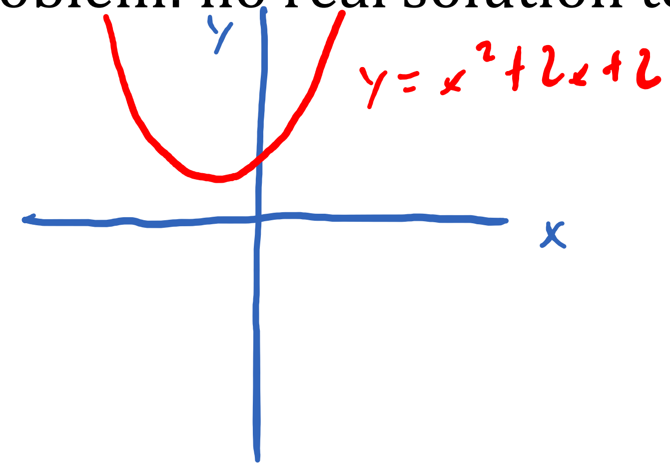
No algebraic closure in real numbers $\mathbb{R}$

- Problem: no real solution to the equation $x^2 + 1 = 0$.



no intersection with x-axis

- Problem: no real solution to the equation $x^2 + 2x + 2 = 0$



no intersection with x-axis

- *Algebraic closure* of the reals means that every polynomial $P(x)$ has to have a real root $P(z) = 0$, but this is not true.

Imagining up a new number

- Let's start by defining a solution to $x^2 + 1 = 0$:

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

- Let $i = \sqrt{-1}$.

Then $i^2 + 1 = (\sqrt{-1})^2 + 1 = -1 + 1 = 0$ ✓

And $(-i)^2 + 1 = (-\sqrt{-1})^2 + 1 = (-1)^2(-1) + 1 = -1 + 1 = 0$ ✓

So $\pm i$ are solutions to $x^2 + 1 = 0$.

Complex numbers

- Since i is a number, we want to be able to add, subtract, multiply, and divide with it, like with real numbers.

Then $2i$, $3i$, $500i$, πi, are all "numbers"

Also $(1-i)$, $2+4i$, etc.

And $\frac{1}{i}$, $\frac{1}{1-i}$, $\frac{2+4i}{1+200i}$.

Even $\sqrt{i}$, e^{i}, i^{i} are all "numbers"

- We call all of these new "numbers" the *complex numbers* $\mathbb{C}$.

Canonical form of complex numbers

- It turns out that every complex number $z \in \mathbb{C}$ can be written simply as $z = a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.

Ex.

$$\frac{1}{1-i} = \frac{(1+i)}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

Ex. $\sqrt{i}$ means a number z s.t. $z^2 = i$.

Try $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \left[\left(\frac{1}{\sqrt{2}}\right)(1+i)\right]^2$$

$$= \frac{1}{2}\cdot(1+i)^2 = \frac{1}{2}(1+2i+i^2) = \frac{1}{2}(1+2i-1) = \frac{1}{2}\cdot 2i = i \quad ✓$$

Aside: $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ also works

Algebraic closure of complex numbers

- $z \in \mathbb{C}$ if $z = a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.
- It turns out that $\mathbb{C}$ is *algebraically closed*, which means that any non-constant polynomial has a root in $\mathbb{C}$.

Ex.

$x^2 + 2x + 2 = 0$

$(x^2 + 2x + 1) + 1 = 0$

$(x+1)^2 + 1 = 0$

$(x+1)^2 = -1$

$x + 1 = \pm i$

$x = -1 \pm i$

or

$x = \dfrac{-2 \pm \sqrt{4 - 8}}{2}$

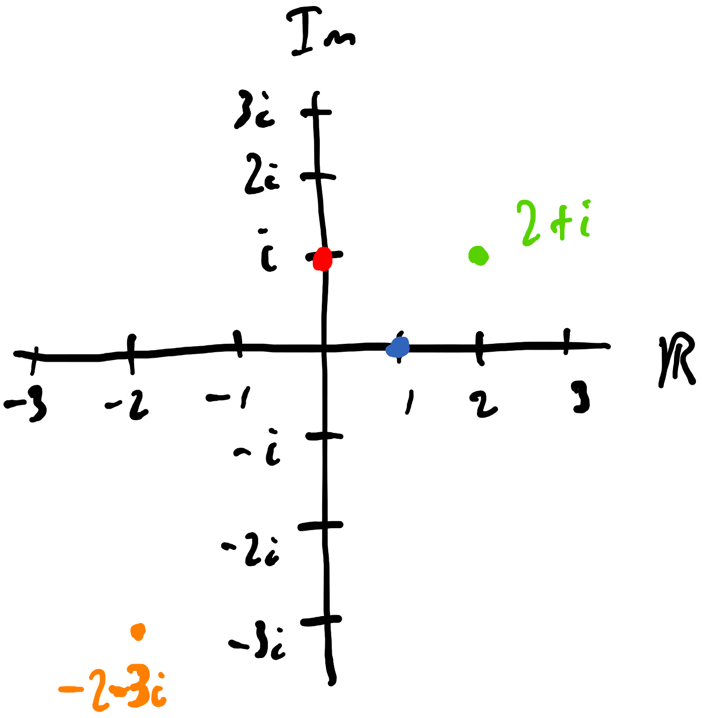
$x = -1 \pm \dfrac{\sqrt{-4}}{2}$

$x = -1 \pm i$

Complex plane

let $i := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- We can use tools from linear algebra to understand $a + bi$.
- Since $a, b \in \mathbb{R}$, we can think of the point $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ as a way to represent $a + bi$.



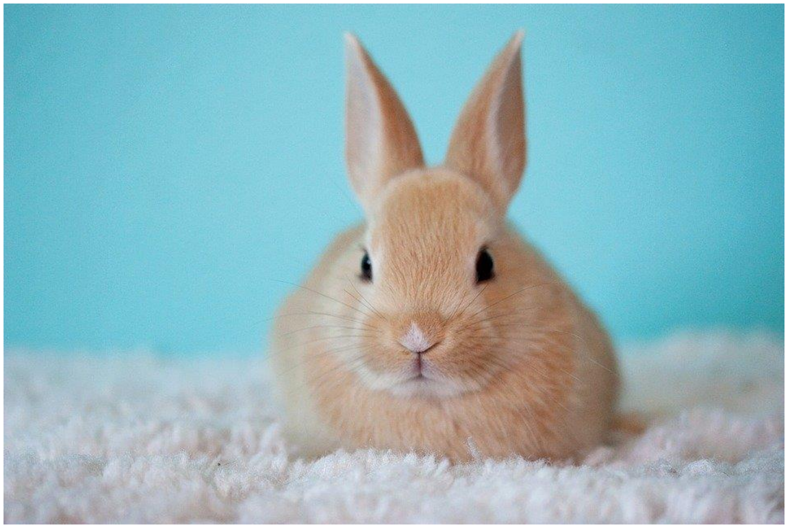
$1 \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$i \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$2 + i \sim 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

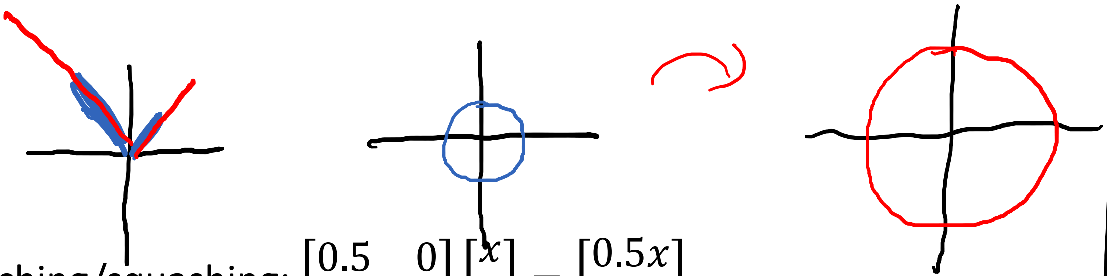
What does this have to do with biology?

- When talking about population sizes “negative” population sizes were considered meaningless because we can’t have negative numbers of animals.
- What does it mean to have “imaginary” numbers of bunnies or birds?

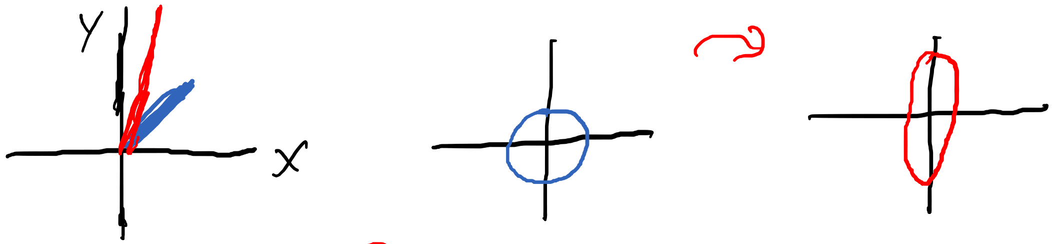


Recall: Matrices are transformations of vectors

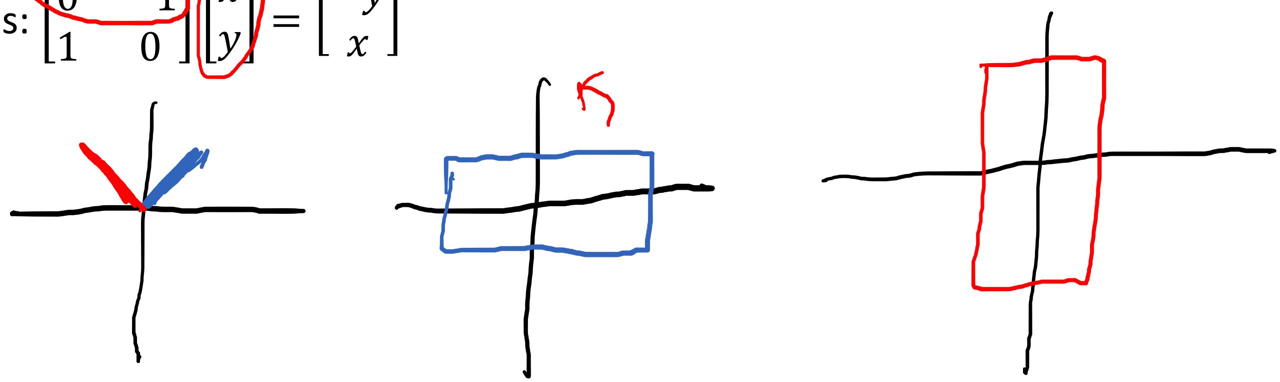
- Scaling operators: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$



- Stretching/squashing: $\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ 2y \end{bmatrix}$



- Rotations: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$



$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A$

$|\lambda I - A| = 0$

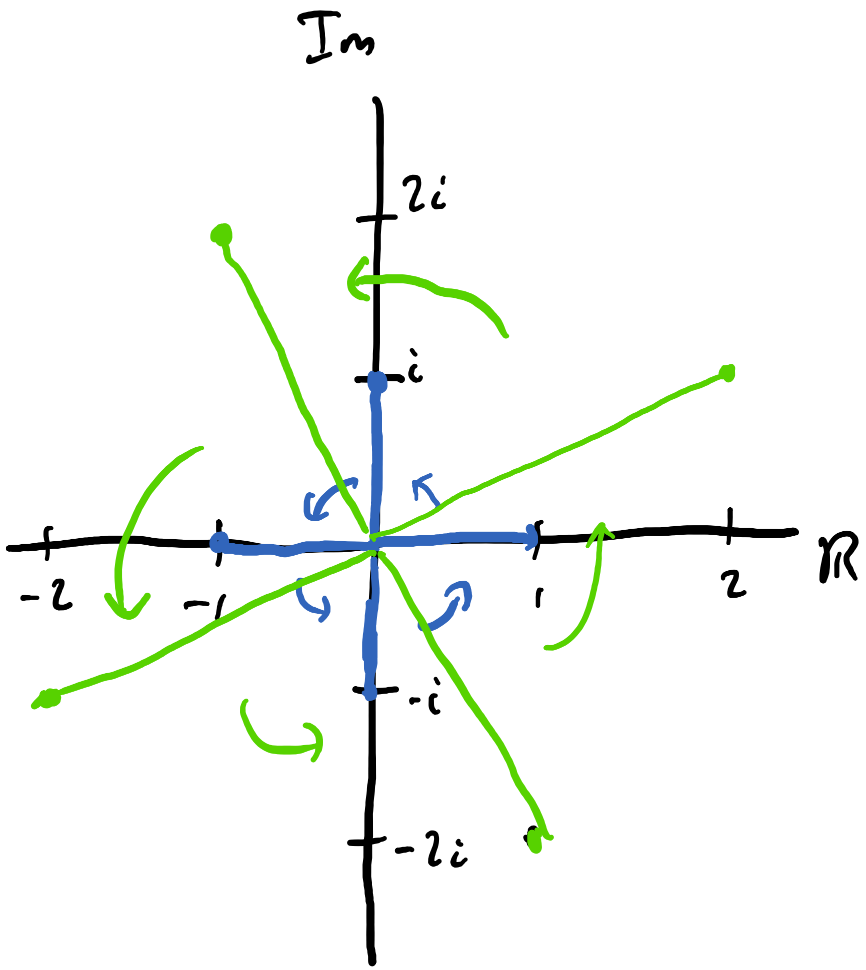
$\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

Imaginary numbers are related to rotation

Multiplication by i = rotation by 90°

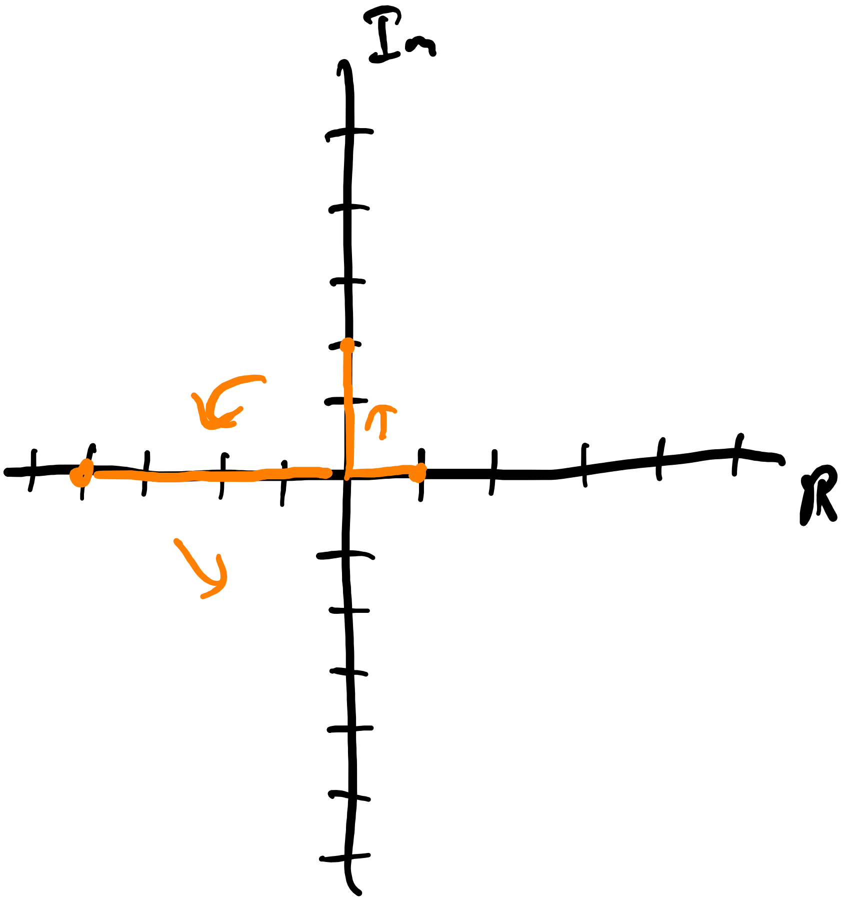


$$i \cdot (1) = i$$
$$i \cdot (i) = -1$$
$$i \cdot (-1) = -i$$
$$i \cdot (-i) = 1$$

$$i \cdot (2+i) = 2i - 1 = -1 + 2i$$
$$i \cdot (-1+2i) = -i - 2 = -2 - i$$
$$i \cdot (-2-i) = -2i + 1 = 1 - 2i$$
$$i \cdot (1-2i) = i + 2 = 2 + i$$

Multiplication by $2i$ = scaling by 2 + rotation by 90°

$2i \cdot 2i = 4i^2 = 4(\sqrt{-1})^2$
$= 4 \cdot -1 = -4$



$2i \cdot 1 = 2i$

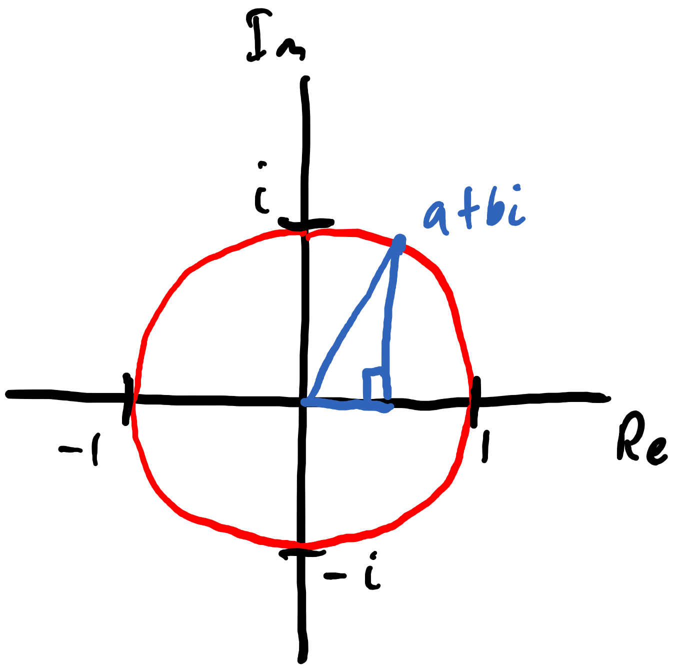
$2i \cdot 2i = -4$

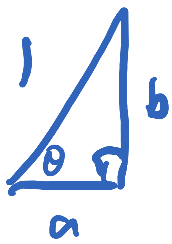
$2i \cdot (-4) = -8i$

$\vdots$

What about other rotation angles?

- We want something to multiply the basis vector 1 by that leaves you with something of length 1 that has the correct angle.
- But multiplying by 1 is the identity.





$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = b$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = a$$

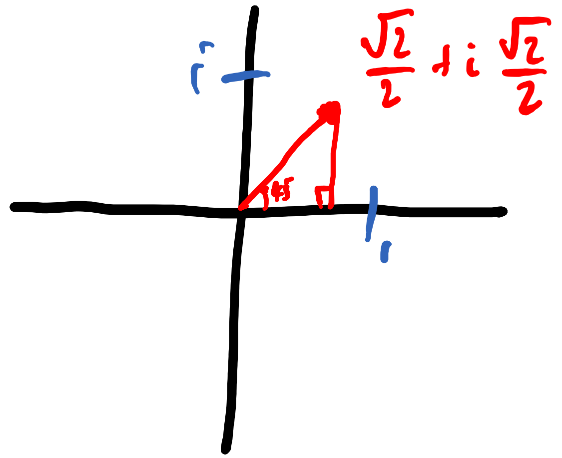
$$a + bi = \cos\theta + i\sin\theta$$

Rotation by θ = multiply by $\cos\theta + i\sin\theta$

Ex. 90° rotation $\cos 90^\circ + i \sin 90^\circ = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = 0 + i = i$

Ex. 45° rotation $\cos 45^\circ + i \sin 45^\circ = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

Recall: $\sqrt{i} = \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}$



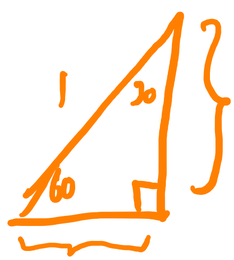
$2\pi = 360^\circ$

60° rotation?

$\cos 60^\circ + i \sin 60^\circ$

$60^\circ = \frac{\pi}{3}$

$\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{2} i$



A: $\frac{1}{2} + \frac{\sqrt{3}}{2} i$

B: $\frac{1}{2} - \frac{\sqrt{3}}{2} i$

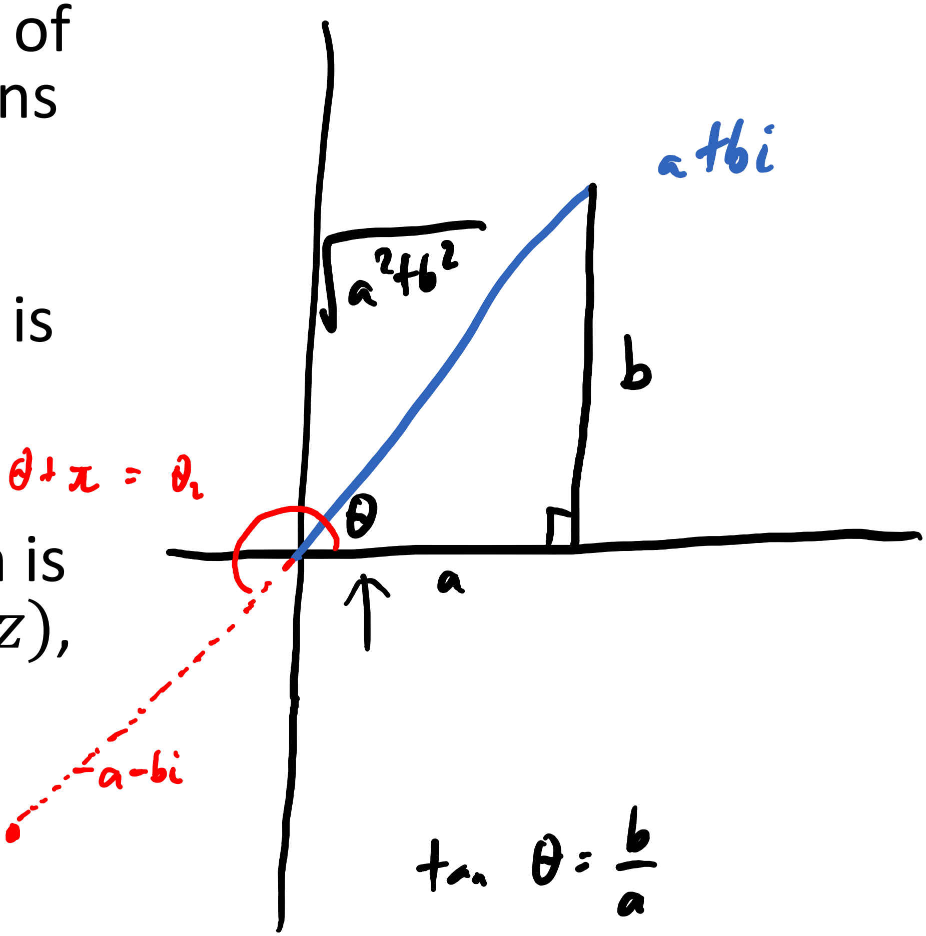
C: $1 + \sqrt{2} i$

D: $1 - \sqrt{2} i$

E: None of the above

Multiplication by $z = a + bi$

- Notice that we can think of all complex multiplications as a rotation and then a scaling.
- The length of the scaling is the *modulus*
$$|z| = \sqrt{a^2 + b^2}$$
- The angle of the rotation is the *argument* $\theta = Arg(z)$, where
$$\frac{a+bi}{|z|} = \cos\theta + i\sin\theta$$



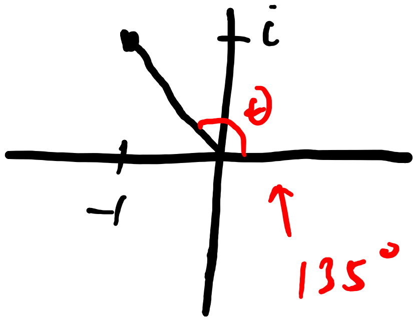
$\tan\theta = \frac{b}{a}$

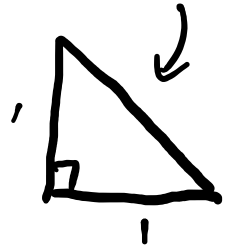
$\tan\theta_2 = \frac{-b}{-a} = \frac{b}{a}$

If $a > 0$, $\theta = \arctan\frac{b}{a}$

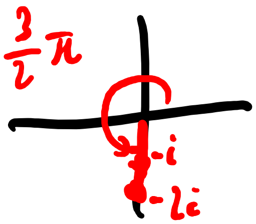
If $a < 0$, $\theta = \arctan\frac{b}{a} + \pi$
(180°)

Try it out





- $|-1+1i| = \sqrt{1+1} = \sqrt{2}$
- $Arg(-1+i) =$ $135° = \frac{135}{180}\pi = \frac{3}{4}\pi$
- $|(-1+i)^2| = |1-2i+i^2| = |(1-2i-1)|$
  $= |-2i| = 2$
- $Arg((-1+i)^2) = Arg(-2i)$
  $= \frac{3}{2}\pi$

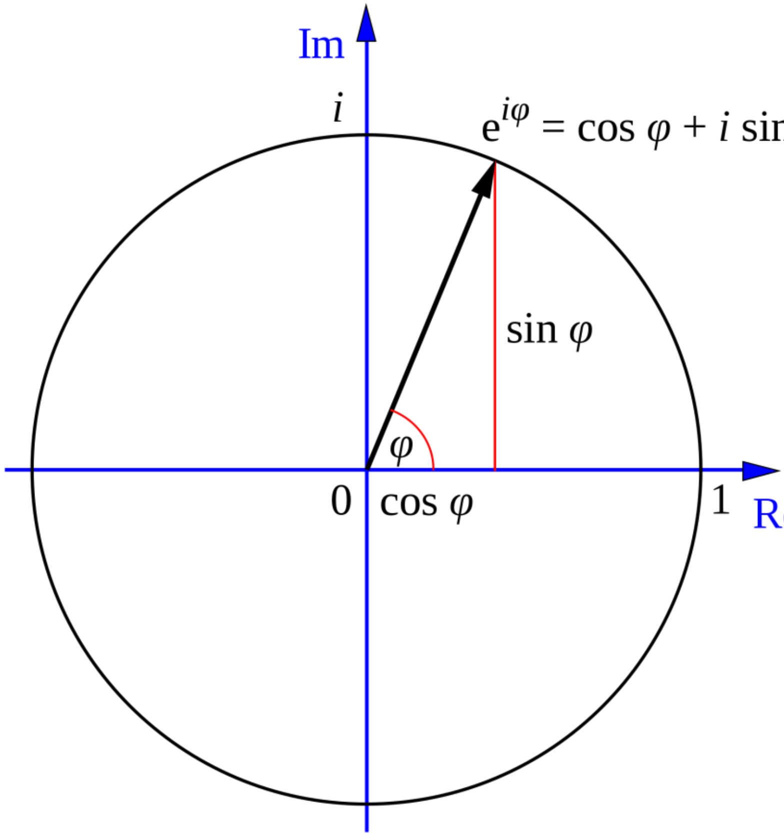


$|i| = 1$
$|-i| = 1$

A: $\frac{3}{2}\pi$
B: $\frac{3}{4}\pi$
C: $\sqrt{2}$
D: 2
E: None of the above

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

- Real powers define exponential growth.
  - $e^0 = 1$
  - $e^1 = e \approx 2.718$
  - $e^2 \approx 7.389$
- Imaginary powers encode rotation around the complex origin.
  - $e^{0i} = 1$
  - $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$
  - $e^{\frac{\pi}{2}i} = i$
  - $e^{\pi i} = -1$



https://en.wikipedia.org/wiki/Euler%27s_formula#/media/File:Euler's_formula.svg

Polar form

- A complex number $z = a + bi$ can be rewritten as a scalar $|z|$ and an angle θ: $z = |z|(\cos\theta + i\sin\theta)$, where $|z| = a^2 + b^2$ and

$$\theta = Arg(z) = \begin{cases} \arctan\frac{b}{a}, & \text{if } a > 0 \\ \arctan\frac{b}{a} + \pi, & \text{if } a < 0 \end{cases}.$$

- Complex exponential $e^z = e^{a+bi} = e^a e^{bi}$
$$= e^a(\cos b + i\sin b)$$
- Thus, $|e^z| = e^a$ and $Arg(e^z) = b$
- So, multiplying by a complex exponential scales by e^a and rotates by an angle b in radians.

Example

$$\boxed{|e^{i\theta}| = 1}$$

Ex 1

$$e^{\frac{\pi}{4}i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

Ex

$$e^{2+\frac{\pi}{4}i} = e^{2}\cdot e^{\frac{\pi}{4}i} = e^{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Try it out: e^{1+i}

$$|e^{1+i}| = |e^{1}\cdot e^{i}| = |e^{1}|\cdot \overset{1}{\overset{\shortparallel}{|e^{i}|}} = e$$

$$\mathrm{Arg}(e^{1+i}) = \underset{0^\circ}{\underset{\shortparallel}{\mathrm{Arg}(e^{1})}} + \mathrm{Arg}(e^{i}) = 1 \text{ rad}$$

$$e^{1+i} = e^{1}(\cos 1 + i\sin 1)$$

A: 0
B: 1
C: e
D: e^2
E: None of the above