Constant coefficient homogeneous higher-order linear ODEs Lecture 9b: 2023-03-13

MAT A35 – Winter 2023 – UTSC

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Recall: linear higher-order ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.

Ex.

$y'' + y' + y = 5$

$\cancel{y'' + y' + y^2 = 5}$

$y''' + \sin(x)y' + x^2y = 5x$

$\cancel{y''' + \sin(y')x + x^2y = 5}$

$\cancel{\dddot{x} + \sin(\dot{x})t + t^2x = 5}$

$\dot{x} = \frac{dx}{dt}$

$y''' + \sin(x)y' + x^2y = 5y$

$\Rightarrow y''' + \sin(x)y' + (x^2 - 5)y = 0$ ✓

A: Linear
B: Nonlinear
C: Both
D: ???
E: None of the above

(In)homogeneous linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is *inhomogeneous*.
  - Note, if nonlinear, then neither definition applies.

$y'' + y' + y = 5$ inhomogeneous

$y'' + y' + y = 0$ homogeneous

$y'' + \sin(x)\, y' + x^2 y = 5y$

$y'' + \sin(x)\, y' + (x^2 - 5)\, y = 0$ homogeneous

Constant coefficient linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $a_i(x) = a_i$ for some constant a_i, then it has constant coefficients
  - Otherwise, is does not have constant coefficients
  - Note, if nonlinear, this terminology does not apply.

$1y'' + 4y' + 2y = 0$ constant coefficients

$y'' + \underline{5x}y' + y = 0$ nonconstant coefficients

$y'' + 5y' + y = \underline{5x}$ constant coefficients

↳ doesn't matter

Try it out: homogeneity and coefficients?

$y' = \frac{dy}{dx}$ ← dep ← ind

$\dot{x} = \frac{dx}{dt}$ ↖ dep ↖ ind.

- $y' + 9y = x^2$ not homogeneous, constant coeff. B
- $y' - \pi y = 0$ A
- $y'' + xy' + y = 0$ C
- $y'' + e^x y' = 3$ D
- $y'' - 2y + y^2 = 5$ E because non linear
- $\ddot{x} + 4\dot{x} = -4x$ $\ddot{x} + 4\dot{x} + 4x = 0$ homog, constant coeff.
- $(\sin x)y'' + e^x y' + y = 0$ C
- $xy'' + y = x^2$ D
- $y'' + 4y + 4 = 0$ $y'' + 4y = -4$ B

A: Homogeneous, constant coefficients
B: Inhomogeneous, constant coefficients
C: Homogeneous, nonconstant coefficients
D: Inhomogeneous, nonconstant coefficients
E: None of the above

Scaling of sols to homogeneous eq

- Let y_1 be a sol. to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Then $c_1 y_1$ is a solution to the same ODE, where c_1 is a constant.

y_1 is a sol to $\underline{a_2 y'' + a_1 y' + a_0 y = 0}$

proof. (order 2)

$$a_2 \cdot \frac{d^2}{dx^2}[c_1 y_1] + a_1 \frac{d}{dx}[c_1 y_1] + a_0 [c_1 y_1] \overset{?}{=} 0$$

$$= c_1 \Big[\underbrace{a_2 \frac{d^2}{dx^2} y_1 + a_1 \frac{d}{dx} y_1 + a_0 y_1}_{} \Big] = 0$$

Ex.

$$y'' + 3y' + 2y = 0$$

Check y_1: $e^{-x} - 3e^{-x} + 2e^{-x} = 0$ ✓

$$y_1 = e^{-x}$$
$$y_1' = -e^{-x}$$
$$y_1'' = e^{-x}$$

Check $5y_1 = 5e^{-x}$

$$\frac{d}{dx}[5e^{-x}] = -5e^{-x}$$
$$\frac{d^2}{dx^2}[5e^{-x}] = 5e^{-x}$$

$$5e^{-x} - 15e^{-x} + 10e^{-x} = 0$$ ✓

Adding sols to homogeneous equation

- Let $y_1(x)$ and $y_2(x)$ be sol. to the homogeneous linear ODE

$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$

- Then $y_1 + y_2$ is a solution to the same ODE.

proof. Expand and collect terms.

Ex. $y'' + 3y' + 2y = 0 \qquad y_1 = e^{-x} \qquad y_2 = e^{-2x}$

Claim: $y_1 + y_2 = e^{-x} + e^{-2x}$

$$[e^{-x} + e^{-2x}]'' + 3[e^{-x} + e^{-2x}]' + 2[e^{-x} + e^{-2x}]$$

$$= e^{-x} + 4e^{-2x} - 3e^{-x} - 6e^{-2x} + 2e^{-x} + 2e^{-2x}$$

$$= 0 \quad \checkmark$$

Main Theorems

- Let $y_1(x), y_2(x), \dots, y_n(x)$ be solutions to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Principal of Superposition: then $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a solution to the same ODE, where c_i are arbitrary constants.
- General solution: If $y_1, \dots, y_n$ are linearly independent, then *all* solutions to the ODE can be written in the form
$$c_1y_1 + c_2y_2 + \cdots + c_ny_n$$
so we call that the general solution to the ODE.

Recall: independent means $c_1y_1 + \cdots + c_ny_n = 0$ only if $c_1 = c_2 = \cdots = c_n = 0$

Constant coefficient homogeneous sol

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$, where a_i are constant.
- We can write a characteristic polynomial
$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$
- If λ is a root of the polynomial (i.e. $p(\lambda) = 0$), then $e^{\lambda x}$ is a solution to the ODE.
- If λ is a root of the polynomial with multiplicity k, then $x^{k-1} e^{\lambda x}$ is a solution to the ODE.
- Note, we will often call λ an eigenvalue of the ODE, for reasons that will become clear later.

Example

- $y'' + 3y' + 2y = 0$

$p(r) = r^2 + 3r + 2$

$p(\lambda) = \lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda = -1, -2$

Thus e^{-x}, e^{-2x}

are sols

$c_1 e^{-x} + c_2 e^{-2x}$

is a general solution

$y'' + 2y' + y = 0$

$p(r) = r^2 + 2r + 1$

$p(\lambda) = \lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\lambda = -1$, multiplicity 2

Then e^{-x}, xe^{-x}

are sols.

$c_1 e^{-x} + c_2 e^{-x}$

are general sols.

Intuitive proof idea

$dy, \quad dx$

$\frac{dy}{dx} \quad \frac{d^2y}{dx^2} \quad \left(\frac{d}{dx}\right)y$

$y'' + 3y' + 2y = 0$

$\frac{d^2}{dx^2}y + 3\frac{d}{dx}y + 2y = 0$

$\left(\frac{d^2}{dx^2} + 3\frac{d}{dx} + 2\right)y = 0$

$\left(\frac{d}{dx}+1\right)\underline{\left(\frac{d}{dx}+2\right)y = 0}$

or

$\left(\frac{d}{dx}+2\right)\underline{\left(\frac{d}{dx}+1\right)y = 0}$

- e^{-2x}
- e^{-x}

Note $\underline{\left(\frac{d}{dx}+\lambda\right)y = 0}$

$\frac{dy}{dx} + \lambda y = 0$

$dy = -\lambda y\, dx$

$\frac{dy}{y} = -\lambda\, dx$

$\ln|y| = -\lambda x + C$ $\quad \boxed{e^{-\lambda x + C}}$

$|y| = Ce^{-\lambda x}$ $\quad C > 0$

$\underline{y = Ce^{-\lambda x}}$

Gen sol: $C_1 e^{-2x} + C_2 e^{-x}$

Try it out

- Which of the following are solutions to

$$y''' - 2y'' - y' + 2y = 0?$$

Char. poly $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$(\lambda - 2)(\lambda^2 - 1) = 0$

$(\lambda - 2)(\lambda + 1)(\lambda - 1) = 0$

$\lambda = -1, 1, 2$

Solutions: $e^{-x}, \quad e^{x}, \quad e^{2x}$

Gen sol: $c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$

A: e^{-x}
B: e^{2x}
C: $e^{-x} + 5e^{x} - 2e^{2x}$
D: All of the above
E: None of the above

Try it out

mult 3, e^{-2x}, xe^{-2x}, x^2e^{-2x}

- Find the general solution to $y'' + 4y' + 4y = 0$.

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)(\lambda+2) = 0$

$\lambda = -2$, multiplicity 2

e^{-2x}, xe^{-2x} are sols

$y_{gen}(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

A: $c_1 e^{-2x}$
B: $c_1 x e^{-2x}$
C: $c_1 e^{-2x} + c_2 x e^{-2x}$
D: All of the above
E: None of the above

- What is the solution to the IVP given $y'' + 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$?

$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

$y'(x) = -2c_1 e^{-2x} - 2c_2 x e^{-2x} + c_2 e^{-2x}$

$1 = y(0) = c_1$

$2 = y'(0) = -2c_1 + c_2$

$2 = -2 + c_2$
$c_2 = 4$

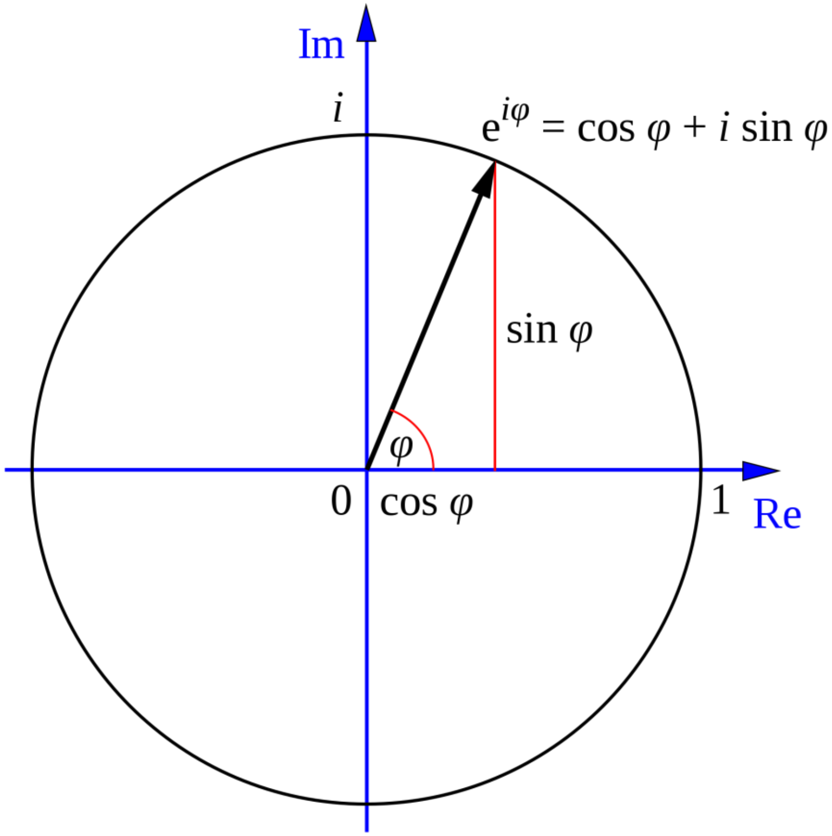
$c_1 = 1$, $c_2 = 4$

$y(x) = e^{-2x} + 4xe^{-2x}$

A: $e^{-2x} + 2xe^{-2x}$
B: $-\frac{1}{3}e^{-2x} + \frac{4}{3}xe^{-2x}$
C: $-\frac{1}{2}e^{-2x} + 2c_2 x e^{-2x}$
D: All of the above
E: None of the above

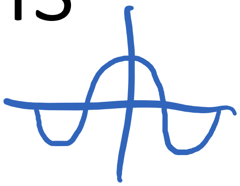
Euler’s Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

- Real powers define exponential growth.
  - $e^0 = 1$
  - $e^1 = e \approx 2.718$
  - $e^2 \approx 7.389$
- Imaginary powers encode rotation around the complex origin.
  - $e^{0i} = 1$
  - $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
  - $e^{\frac{\pi}{2}i} = i$
  - $e^{\pi i} = -1$



https://en.wikipedia.org/wiki/Euler%27s_formula#/media/File:Euler's_formula.svg

Complex roots → Real solutions



- Consider the equation $y'' + y = 0$
- Use $e^{ix} = \cos x + i \sin x$

$\lambda^2 + 1 = 0$

$\lambda = \pm\sqrt{-1} = \pm i$

$(\lambda + i)(\lambda - i) = 0$

$\lambda = i, -i$

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos(-x) + i \sin(-x)$

$e^{-ix} = \cos x - i \sin x$

$y = c_1 e^{ix} + c_2 e^{-ix}$ $\Leftarrow$

$y = c_1 \cos x + i c_1 \sin x + c_2 \cos x - i c_2 \sin x$

$y = \underbrace{(c_1 + c_2)}_{\hat{c}_1} \cos x + \underbrace{i(c_1 - c_2)}_{\hat{c}_2} \sin x$

complex

Or $y = \hat{c}_1 \cos x + \hat{c}_2 \sin x$ $\Leftarrow$

Another example

- $y'' + 2y' + 5y = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$y = c_1 e^{(-1+2i)x} + c_2 e^{(-1-2i)x}$

$e^{(-1+2i)x} = e^{-x} e^{2ix}$

$= e^{-x}[\cos 2x + i \sin 2x]$

$e^{(-1-2i)x} = e^{-x}[\cos 2x - i \sin 2x]$

So $\underline{e^{-x} \cos 2x}$ and $\underline{e^{-x} \sin 2x}$ are ind. sols.

Or

$y = \hat{c}_1 e^{-x} \cos(2x) + \hat{c}_2 e^{-x} \sin(2x)$

Or

$y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

Complex roots with real coefficients

- Complex roots of a real polynomial always come in pairs $a \pm ib$.
- If a characteristic equation of an ODE has roots $a \pm ib$, then has complex solutions $e^{(a+ib)x}$ and $e^{(a-ib)x}$.
- Alternately, it has real solutions $e^{ax} \sin bx$ and $e^{ax} \cos bx$

Ex.

$$y'' - 2y' + 10y = 0$$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm \sqrt{-9} = 1 \pm 3i$$

$$y = c_1 e^x \sin 3x + c_2 e^x \cos 3x$$

Try it out

- Let $y'' + 4y' + 29y = 0$.
- Which of the following are solutions to the ODE?

$\lambda^2 + 4\lambda + 29 = 0$

$\lambda^2 + 4\lambda + 4 + 25 = 0$

$(\lambda+2)^2 = -25$

$\lambda + 2 = \pm 5i$

$\lambda = -2 \pm 5i$

$e^{(-2+5i)x}, \quad e^{(-2-5i)x}$

$e^{-2x} \cos 5x, \quad e^{-2x} \sin 5x$

- What about real solutions?

A: $e^{(-2+5i)x} + 4e^{(-2-5i)x}$

B: $-\pi e^{-2x} e^{5ix} = -\pi e^{-2x+5ix}$

only real $\longrightarrow$ C: $e^{-2x} \cos 5x$

D: All of the above complex

E: None of the above

Repeated complex eigenvalues of ODE

- Like repeated real roots, if $a \pm bi$ have multiplicity k, then $x^{k-1}e^{ax}\cos bx$ and $x^{k-1}e^{ax}\sin bx$ are solutions.

Ex.

$$y'''' + 4y'' + 4y = 0$$

$$\lambda^4 + 4\lambda^2 + 4 = 0$$

$$(\lambda^2+2)^2 = 0 \rightarrow (\lambda + i\sqrt{2})^2(\lambda - i\sqrt{2})^2 = 0$$

$$\lambda^2 = -2$$

$$\lambda = \pm i\sqrt{2}, \text{ mult } 2$$

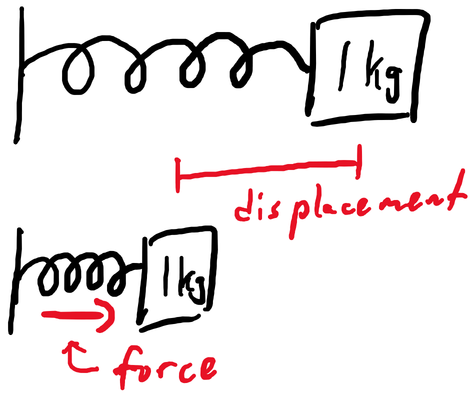
$$\Rightarrow y = c_1 \cos(\sqrt{2}x) + c_2 x\cos(\sqrt{2}x) + c_3 \sin(\sqrt{2}x) + c_4 x \sin(\sqrt{2}x)$$

Summary

- To solve a linear nth-order homogeneous ODE
$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$
- Construct the characteristic equation
$$a_n \lambda^n + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$
- The n roots (counting multiplicity) of the characteristic equation are either real or come in complex conjugate pairs.
- If λ is a (real or complex) root of multiplicity k, then $e^{\lambda x}, xe^{\lambda x}, \ldots x^{k-1}e^{\lambda x}$ are linearly independent solutions.
- If $\lambda = a \pm ib$ is a conjugate pair of complex roots, each of multiplicity k, then $e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{k-1}e^{ax}\cos bx$ and $e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{k-1}e^{ax}\sin bx$ are $2k$ linearly independent solutions.

Application: mass-spring system

- A spring acts on an attached 1kg object with force -4 Newtons/meter times the displacement in meters.
- Let y be the displacement of the object at time x, and y' is its velocity.
- By Newton's 2nd law, $F = ma$, where F is force, m is mass, and $a = y''$ is acceleration.

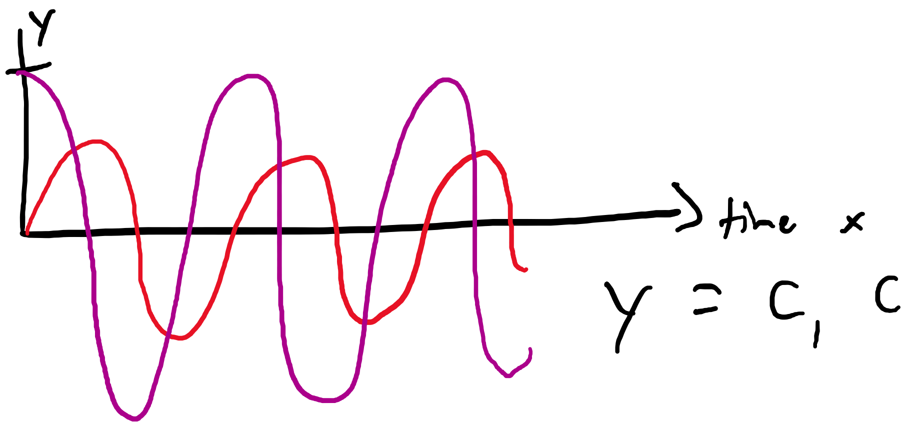


$y'' = -4y$

$y'' + 4y = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$



$y = c_1 \cos 2x + c_2 \sin 2x$

Initial Value Problem

- $y'' + 4y = 0$, where $y'(0) = 0$, $y(0) = 10$.

$y = c_1 \cos 2x + c_2 \sin 2x$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x$

$10 = y(0) = c_1 \cdot 1 + c_2 \cdot 0$

$\Rightarrow c_1 = 10$

$0 = y'(0) = -2c_1 \cdot 0 + 2c_2 \cdot 1$

$\Rightarrow c_2 = 0$

$\Rightarrow y = 10 \cos 2x$