Constant coefficient homogeneous higher-order linear ODEs Lecture 9b: 2023-03-13

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Recall: linear higher-order ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.

A: Linear
B: Nonlinear
C: Both
D: ???
E: None of the above

(In)homogeneous linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is *inhomogeneous*.
  - Note, if nonlinear, then neither definition applies.

Constant coefficient linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $a_i(x) = a_i$ for some constant a_i, then it has constant coefficients
  - Otherwise, is does not have constant coefficients
  - Note, if nonlinear, this terminology does not apply.

Try it out: homogeneity and coefficients?

- $y' + 9y = x^2$
- $y' - \pi y = 0$
- $y'' + xy' + y = 0$
- $y'' + e^x y' = 3$
- $y'' - 2y + y^2 = 5$
- $\ddot{x} + 4\dot{x} = -4x$
- $(\sin x)y'' + e^x y' + y = 0$
- $xy'' + y = x^2$
- $y'' + 4y + 4 = 0$

A: Homogeneous, constant coefficients
B: Inhomogeneous, constant coefficients
C: Homogeneous, nonconstant coefficients
D: Inhomogeneous, nonconstant coefficients
E: None of the above

Scaling of sols to homogeneous eq

- Let y_1 be a sol. to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Then $c_1 y_1$ is a solution to the same ODE, where c_1 is a constant.

Adding sols to homogeneous equation

- Let $y_1(x)$ and $y_2(x)$ be sol. to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Then $y_1 + y_2$ is a solution to the same ODE.

Main Theorems

- Let $y_1(x), y_2(x), \dots, y_n(x)$ be solutions to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Principal of Superposition: then $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a solution to the same ODE, where c_i are arbitrary constants.
- General solution: If $y_1, \dots, y_n$ are linearly independent, then *all* solutions to the ODE can be written in the form
$$c_1y_1 + c_2y_2 + \cdots + c_ny_n$$
so we call that the general solution to the ODE.

Constant coefficient homogeneous sol

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$, where a_i are constant.
- We can write a characteristic polynomial
$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$
- If λ is a root of the polynomial (i.e. $p(\lambda) = 0$), then $e^{\lambda x}$ is a solution to the ODE.
- If λ is a root of the polynomial with multiplicity k, then $x^{k-1} e^{\lambda x}$ is a solution to the ODE.
- Note, we will often call λ an eigenvalue of the ODE, for reasons that will become clear later.

Example

- $y'' + 3y' + 2y = 0$

$y'' + 2y' + y = 0$

Intuitive proof idea

Try it out

- Which of the following are solutions to
$$y''' - 2y'' - y' + 2y = 0?$$

A: e^{-x}
B: e^{2x}
C: $e^{-x} + 5e^{x} - 2e^{2x}$
D: All of the above
E: None of the above

Try it out

- Find the general solution to $y'' + 4y' + 4y = 0$.

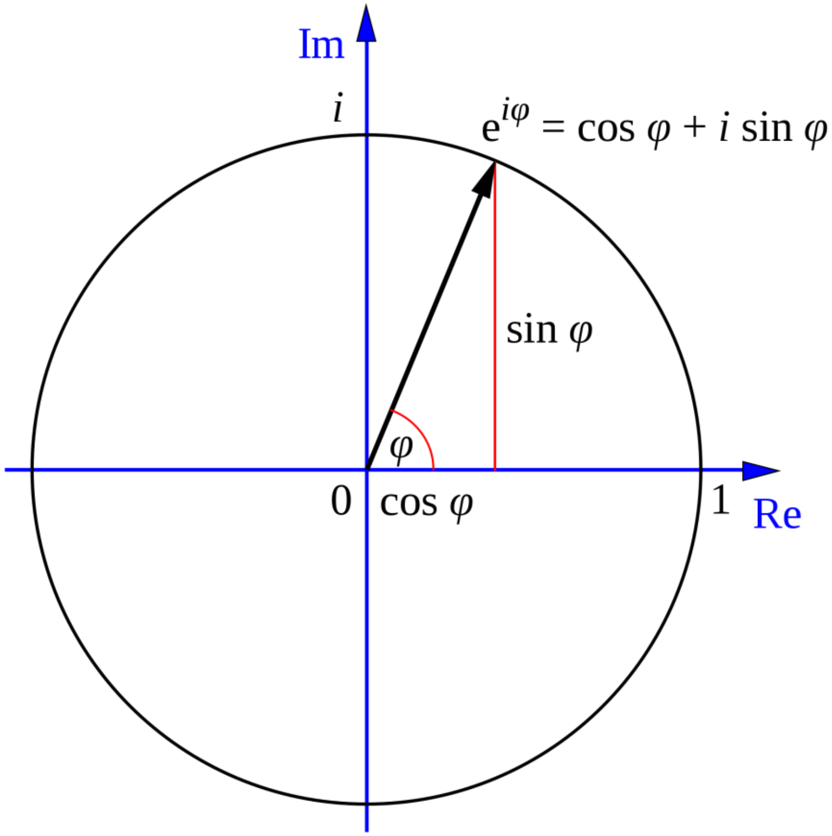
A: $c_1 e^{-2x}$
B: $c_1 x e^{-2x}$
C: $c_1 e^{-2x} + c_2 x e^{-2x}$
D: All of the above
E: None of the above

- What is the solution to the IVP given $y'' + 4y' + 4y = 0,\ y(0) = 1,\ y'(0) = 2$?

A: $e^{-2x} + 2xe^{-2x}$
B: $-\frac{1}{3}e^{-2x} + \frac{4}{3}xe^{-2x}$
C: $-\frac{1}{2}e^{-2x} + 2c_2 xe^{-2x}$
D: All of the above
E: None of the above

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

- Real powers define exponential growth.
  - $e^0 = 1$
  - $e^1 = e \approx 2.718$
  - $e^2 \approx 7.389$
- Imaginary powers encode rotation around the complex origin.
  - $e^{0i} = 1$
  - $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
  - $e^{\frac{\pi}{2}i} = i$
  - $e^{\pi i} = -1$



https://en.wikipedia.org/wiki/Euler%27s_formula#/media/File:Euler's_formula.svg

Complex roots → Real solutions

- Consider the equation $y'' + y = 0$
- Use $e^{ix} = \cos x + i \sin x$

Another example

- $y'' + 2y' + 5y = 0$

Complex roots with real coefficients

- Complex roots of a real polynomial always come in pairs $a \pm ib$.
- If a characteristic equation of an ODE has roots $a \pm ib$, then has complex solutions $e^{(a+ib)x}$ and $e^{(a-ib)x}$.
- Alternately, it has real solutions $e^{ax} \sin bx$ and $e^{ax} \cos bx$

Try it out

- Let $y'' + 4y' + 29y = 0$.
- Which of the following are solutions to the ODE?

- What about real solutions?

A: $e^{(-2+5i)x} + 4e^{(-2-5i)x}$
B: $-\pi e^{-2x} e^{5ix}$
C: $e^{-2x} \cos 5x$
D: All of the above
E: None of the above

Repeated complex eigenvalues of ODE

- Like repeated real roots, if $a \pm bi$ have multiplicity k, then $x^{k-1}e^{ax}\cos bx$ and $x^{k-1}e^{ax}\sin bx$ are solutions.

Summary

- To solve a linear nth-order homogeneous ODE
$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$
- Construct the characteristic equation
$$a_n \lambda^n + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$
- The n roots (counting multiplicity) of the characteristic equation are either real or come in complex conjugate pairs.
- If λ is a (real or complex) root of multiplicity k, then $e^{\lambda x}, xe^{\lambda x}, \ldots x^{k-1}e^{\lambda x}$ are linearly independent solutions.
- If $\lambda = a \pm ib$ is a conjugate pair of complex roots, each of multiplicity k, then $e^{ax}\cos bx\,, xe^{ax}\cos bx\,, \ldots, x^{k-1}e^{ax}\cos bx$ and $e^{ax}\sin bx\,, xe^{ax}\sin bx\,, \ldots, x^{k-1}e^{ax}\sin bx$ are $2k$ linearly independent solutions.

Application: mass-spring system

- A spring acts on an attached 1kg object with force -4 Newtons/meter times the displacement in meters.
- Let y be the displacement of the object at time x, and y' is its velocity.
- By Newton's 2nd law, $F = ma$, where F is force, m is mass, and $a = y''$ is acceleration.

Initial Value Problem

- $y'' + 4y = 0$, where $y'(0) = 0$, $y(0) = 10$.