# Nonhomogeneous constant coefficient ODEs Lecture 9c: 2023-03-16

MAT A35 – Winter 2023 – UTSC

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# (In)homogeneous constant coefficient linear ODEs

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$, where $a_i$ are constant coefficients and $q(x)$ is a functions of $x$.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is *inhomogeneous*.

# Solution to inhomogeneous problems

- Consider the inhomogeneous equation
$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$$
- The associated homogeneous equation (which we know how to solve) is:
$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$
- If $y_p$ is a any "particular" solution to the inhomogeneous equation, and $y_h$ is the general solution to the associated homogeneous equation, then $y = y_p + y_g$ is the general solution to the inhomogeneous equation.

# Example

- $y'' + 3y' + 2y = 6$

# Example

- $y'' + 3y' + 2y = e^{-3x}$

# Method of undetermined coefficients

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$
- Notice that whatever we guess for the particular solution $y_p$ we have to take derivatives of it. A reasonable “Ansatz”, guess, is $y_p$ will “look like” the derivatives of $q(x)$ but with different coefficients.

# Try it out: guess an Ansatz

- $q(x) = e^x + e^{2x}$

A: $Ae^x$
B: $Ae^{2x}$
C: $Ae^x + Be^{2x}$
D: $Ae^x + Be^{2x} + C$
E: None of the above

- $q(x) = 3x^2 + \sin x$

A: $Ax^2 + B\sin x$
B: $Ax^2 + B\sin x + C\cos x$
C: $Ax^2 + Bx + C + D\sin x$
D: $Ax^2 + Bx + C + D\sin x + E\cos x$
E: None of the above

- $q(x) = \frac{1}{x}$

A: $A\ln x + B$
B: $\frac{A}{x} + B$
C: $\frac{A}{x} + \frac{B}{x^2} + D$
D: $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + D$
E: None of the above

# Ansatz-homogeneous solution collisions

- What if your Ansatz looks like one of the homogeneous solutions?
- Then just like with repeated roots, will need to add an "$x$“.

# Try it out: guess an Ansatz $y_p$

- $y'' + 3y' + 2y = e^x + e^{2x}$

A: $Ae^x + Be^{2x}$
B: $Axe^x + Be^{2x}$
C: $Ae^x + Bxe^{2x}$
D: $Axe^x + Bxe^{2x}$
E: None of the above

- $y'' - y = e^x + e^{2x}$

A: $Ae^x + Be^{2x}$
B: $Axe^x + Be^{2x}$
C: $Ae^x + Bxe^{2x}$
D: $Axe^x + Bxe^{2x}$
E: None of the above

- $y'' + y = \sin x$

A: $A \sin x$
B: $A \sin x + B \cos x$
C: $Ax \sin x + B \cos x$
D: $Ax \sin x + Bx \cos x$
E: None of the above

# Example

- $y' + 2y = x^2, y(0) = 1$

# Summary

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$
- We can compute the homogeneous solution by looking at roots of the characteristic polynomial $a_n \lambda^n + \cdots + a_1 \lambda + a_0 = 0$, and independent solutions will be of the form $e^{\lambda x}$ or $e^{Re(\lambda)x} \cos(Im(\lambda)x)$ and $e^{Re(\lambda)x} \sin(Im(\lambda)x)$.
- We can often guess a particular solution by using an Ansatz with undetermined coefficients that looks like the derivatives of $q(x)$. We can then solve for the coefficients.
- The general solution is then given by the homogeneous solution plus any particular solution.
- We can solve an initial value problem by plugging those values back into the general solution.