MATA35H3 Winter 2023 Sample questions Student ID:________________________

Computer and Mathematical Sciences **University of Toronto at Scarborough**

Duration: Undefined **Aids Allowed: None**

- Write your Student ID at the top of this page.
- This question sample has 14 questions and 8 pages.
- In order to receive credit, you must show your work.
- Simplify your answers whenever possible.
- Calculators, phones, and other electronic devices are not permitted.

Question	Points
1	0
2	0
3	0
⋮	0
⋮	0
12	0
13	0
14	0
Total	0

Formulas that may be useful:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{cosx}$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x \qquad (\arctan x)' = \frac{1}{1+x^2}$$

$$\sin^2 x + \cos^2 x = 1 \qquad \sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

Integration by parts: $\int u\,dv = uv - \int vdu$

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1. **0 points.** Solve the given system of linear equations (Hint: use Gauss-Jordan elimination):

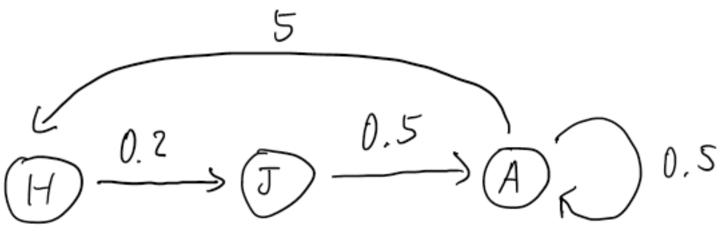
$$\begin{cases} x+y+z+w=5 \\ x+z+w=6 \\ y+z+w=4 \\ x+z=3 \end{cases}$$

2. **0 points.** Find a cubic approximation near 0 for the following function using the first 4 terms of its Maclaurin Series (Taylor Series at 0).

$$f(x) = \frac{1+x}{(1-x)^2}.$$

$$\textbf{Hint: } \frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

3. **0 points.** A species of bird has three life stages, hatching (H), juvenile (J), and adult (A). After some study, you build a Leslie Age-Structured model capturing the yearly change in the population with the following Leslie diagram:

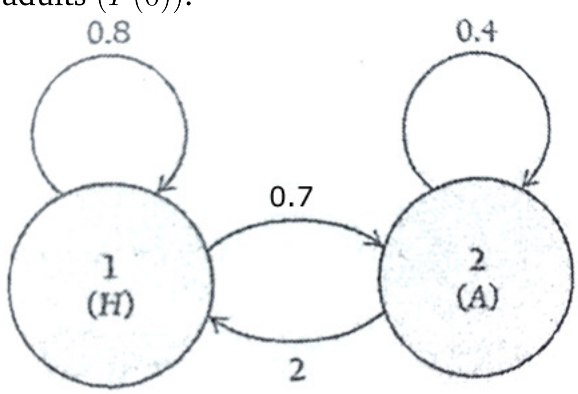


(a) Write down the Leslie matrix for this model. Given this diagram, what is the probability that a juvenile survives to become an adult?
Hint: the determinant of the Leslie matrix is 0.5. You may wish to check you have the correct Leslie matrix before moving on.

(b) If right now, the population consists of 200 hatchlings, 20 juveniles, and 60 adults, estimate the population one year into the future. Estimate the population two years into the future.

(c) Using the same model, estimate the population one year ago.

4. **0 points.** A population of birds with the Leslie diagram below has 120 hatchlings and 80 adults $(P(0))$.



(a) Write the Leslie matrix.

(b) Estimate the opulation of hatchlings after 20 breeding seasons $(P(20))$.

(c) What is the long-term growth rate of the given population?

5. **0 points.** Solve one of the following two problems in as much generality as possible. If you solve both problems, you will receive the higher of the two scores.

(a) Solve the initial value problem: $y' = \frac{x}{e^x \sin y}$, where $y(0) = 0$.

(b) Give an implicit solution to the following initial value problem:
$\left(\frac{2x}{(x+y)(x-y)} + 2x\right) dx + \left(\frac{-2y}{(x+y)(x-y)} - 2y\right) dy = 0$, where $y(1) = 0$.

6. **0 points.** Solve one of the following two problems in as much generality as possible. If you solve both problems, you will receive the higher of the two scores.

(a) Find the general solution: $y'' + 2y' + 5y = x + e^x$.

(b) Find the solution to the following first-order linear system initial value problem (give your answer as $x(t) = \dots$ and $y(t) = \dots$):

$$\begin{aligned} \dot{x} &= x - 3y, \quad x(0) = 5 \\ \dot{y} &= x + 5y, \quad y(0) = -3 \end{aligned}$$

7. **0 points.** Solve the given differential equations:

(a) $x\frac{dy}{dx} + y = x^2y^2, \quad y(1) = 1$

(b) $y' + 4xy = x$

8. **0 points.**

(a) Find and classify all critical points of $f(x, y, z) = x^2 + y^2 + z^2 + 2xyz$

(b) A rectangular box, open at the top, is to hold 256 cubic centimetres of sand. Find the dimensions for which the surface area (bottom and four sides) is minimized.

9. **0 points.** Solve **BOTH** of the following problems.

(a) Let $f(x, y) = \frac{1}{3}x^3 - \frac{1}{2}y^2 - xy$.
Find all the (real) critical points of f. Identify whether the critical points found above are relative minimums, relative maximums, or saddle points.

(b) Consider the following autonomous ODE system:

$$\frac{dx}{dt} = x^2 - y \frac{dy}{dt} = -x - y$$

Find the coordinates of the real equilibria. Classify the equilibria and determine their stability.

10. **0 points.** Find and classify all critical points for the given non-linear system. Draw the phase portrait of the given system. Carefully indicate the direction of motion with increasing t. (You'll need to find eigenvectors as well as eigenvalues)

$$\begin{cases} x' = xy \\ y' = x + 2y - 8 \end{cases}$$

11. **0 points.** Given the system of linear differential equations $\begin{cases} x' = 3x - 2y \\ y' = 4x - y \end{cases}$

(a) Solve the given system of linear differential equations

(b) Indicate the character and stability of the critical point.

(c) Draw the family of solution curves and show the direction of motion with increasing t.

12. **0 points.** The autonomous equation $y'(t) = y(t) \cdot (y^2(t) - 4)$ describes the change in the number of people in a confined population who have contracted contagious, but not a fatal disease. Draw the family of solution curves and explain what happens to the population if initially

(a) Only one member of the population have a disease?

(b) Three members of the population have the disease?

13. **0 points.** A 2000 gal tank initially contains 200 lbs of salt dissolved in 500 gal of water. Pure water is pumped into the tank at a rate of 5 gal per min, while the well-stireed mixture is drawn off at a rate of 2 gal per min. The process stops when the tank is full. How much salt is left in the tank when the tank is full? Carefully indicate each step of mathematical modelling: physical model, math model (derivation of the equation), math analysis (solution), results.

14. **0 points.** You are an epidemiologist from the University of Toronto who was recently tasked with understanding the Cooties. To this end, you observe an outbreak at a local primary school, and come to several conclusions:

- The disease has no adverse side-effects. In fact, if it weren't for the children loudly proclaiming they or another person caught the Cooties, you wouldn't be able to determine their infection status.
- No one is immune from the Cooties.
- Cooties are transmitted at a rate proportional to both the number of Susceptible individuals and the number of Infected individuals.
- Infected individuals seem to recover spontaneously at a high random rate.

After weeks of painstaking observation, you gather enough information to build a two-compartment model:

- There are a total of $N = 100 = S(t) + I(t)$ kids at the school.
- Kids are either Susceptible (S) or Infected (I)
- 0.001 SI Susceptible individuals become infected each day.
- Infected individuals recover and become Susceptible again at a rate of 10% per day.

(a) Draw a 2-compartment model for S and I.

(b) Convert the model to a one-variable model counting only infected individuals.

(c) Find the equilibrium values.

(d) Classify the equilibria by mathematical stability.

(e) What is the long-term behavior of the system? Explain in words what happens.