MATA35 - Quiz 2 - Practice

Name: Solutions Student ID: N/A

Problem 1 [5pts]. Solve each of the following problems. If the answer is undefined, state so explicitly.

(a) $\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ undefined (cannot multiply 2×3 by 2×3)

(c) $\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & -1 \\ 6 & -2 & 2 \\ -6 & 2 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -1 & 5 \\ 1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ undefined (cannot add 2×3 by 3×1)

(e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ (because multiplication by identity)

Problem 2 [4pts]. Find the determinant of the following matrix. Does the matrix have a multiplicative inverse? Why?

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = 3 \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 3 \cdot (-2) + (-1) - 1 = \boxed{-8}$$

Or

$$\left[\begin{array}{ccc|cc} 3 & -1 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$= -1 - (6 + 1) = -8$$

Yes, the matrix has a multiplicative inverse because the determinant is nonzero.

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Problem 3 [8pts]. Find the determinant, multiplicative inverse, all eigenvalues, and all eigenvectors for the following matrix. Show your work.

$A = \begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}$

$$\det\begin{pmatrix} -5 & 4 \\ -4 & 5 \end{pmatrix} = -25 + 16 = -9$$

Find inverse: $\left[\begin{array}{cc|cc} -5 & 4 & 1 & 0 \\ -4 & 5 & 0 & 1 \end{array}\right]$

$R_1 \leftarrow R_1/-5$, $R_2 \leftarrow R_2/-4$

$$\left[\begin{array}{cc|cc} 1 & -4/5 & -\frac{1}{5} & 0 \\ 1 & -5/4 & 0 & -\frac{1}{4} \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{cc|cc} 1 & -4/5 & -1/5 & 0 \\ 0 & -9/20 & 1/5 & -\frac{1}{4} \end{array}\right]$$

$R_2 \leftarrow R_2 \cdot \left(-\frac{20}{9}\right)$

$$\left[\begin{array}{cc|cc} 1 & -4/5 & -1/5 & 0 \\ 0 & 1 & -\frac{4}{9} & \frac{5}{9} \end{array}\right]$$

$R_1 \leftarrow R_1 + \frac{4}{5} \cdot R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{5}{9} & \frac{4}{9} \\ 0 & 1 & -\frac{4}{9} & \frac{5}{9} \end{array}\right]$$

Inverse: $\begin{bmatrix} -5/9 & 4/9 \\ -4/9 & 5/9 \end{bmatrix}$

Eigenvalues: $|A - \lambda I| = \begin{vmatrix} -5-\lambda & 4 \\ -4 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow -25 + \lambda^2 + 16 = 0$

$\lambda^2 - 9 = 0$

$(\lambda + 3)(\lambda - 3) = 0$

$$\boxed{\lambda_1 = 3, \quad \lambda_2 = -3}$$

Find eigenvectors:

$\lambda_1 = 3$

$$\begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

$-5x + 4y = 3x$

$4y = 8x$

$y = 2x$

$$v_1 = \begin{bmatrix} x \\ 2x \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find eigenvectors:

$\lambda_2 = -3$

$$\begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3x \\ -3y \end{bmatrix}$$

$-5x + 4y = -3x$

$4y = 2x$

$x = 2y$

$$v_2 = \begin{bmatrix} 2y \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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Problem 4 [8pts]. As a backyard ecologist, you have been keeping track (as best you can) of the dandelion population in your yard. Each year, dandelions pop up from two sources:

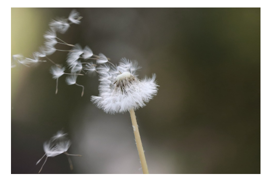
1. New seedlings (S): these are new grown plants that grow from the tens of thousands of seeds that are released.
2. Old taproots (T): these are old plants whose root systems have survived the winter and have come back as perennials.

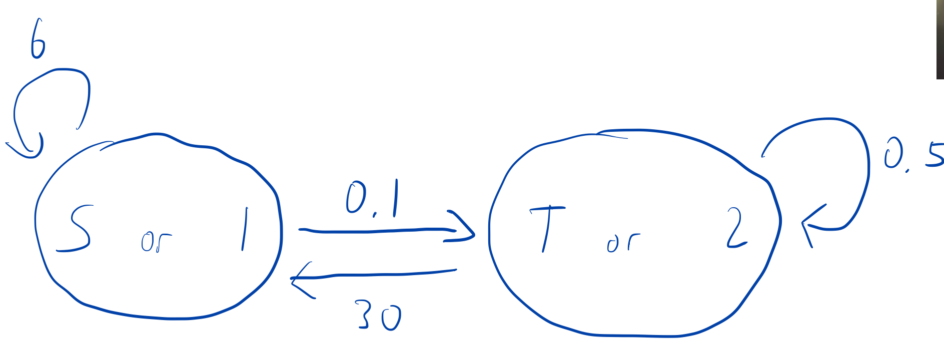
Because of the short time to seed, even new seedlings will produce seeds over the course of a year, so both new seedlings and old taproots will produce seeds. After careful counting, you decide to build a model off the following observations.

- Each year, a seedling has a 0.1 probability of surviving to the next year.
- Each year, a seedling produces 6 new seedlings for the following year.
- Each year, old taproots have a 0.5 probability of surviving to the new year.
- Each year, old taproots produce 30 new seedlings for the following year.

Answer the following questions:

(a) Write a Leslie matrix and Leslie diagram for this age-structured model.





$$L = \begin{bmatrix} 6 & 30 \\ 0.1 & 0.5 \end{bmatrix}$$

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(b) Suppose your yard starts with 35 seedlings and 6 taproots this year. How many seedlings and taproots will you have after 10 years time? (you do not need to evaluate powers and exponents and can leave your answer unreduced)
Hint: the eigenvalues of your Leslie matrix should be 6.5 and 0.

$\lambda_1 = 6.5$ $\begin{bmatrix} 6 & 30 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6.5x \\ 6.5y \end{bmatrix}$

$6x + 30y = 6.5x$

$30y = 0.5x$

$60y = x$

$v_1 = \begin{bmatrix} 60 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$ $\begin{bmatrix} 6 & 30 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$6x + 30y = 0$

$6x = -30y$

$x = -5y$

$v_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 35 \\ 6 \end{bmatrix} = a\begin{bmatrix} 60 \\ 1 \end{bmatrix} + b\begin{bmatrix} -5 \\ 1 \end{bmatrix}$

$\begin{cases} 35 = 60a - 5b \\ 6 = a + b \end{cases}$

$b = 6 - a$

$35 = 60a - 30 + 5a$

$65 = 65a$

$a = 1$

$b = 5$

$L^{10}\begin{bmatrix} 35 \\ 6 \end{bmatrix} = L^{10}\begin{bmatrix} 60 \\ 1 \end{bmatrix} + 5L^{10}\begin{bmatrix} -5 \\ 1 \end{bmatrix}$

$= 6.5^{10} \cdot \begin{bmatrix} 60 \\ 1 \end{bmatrix} + 0.$

$\boxed{6.5^{10} \cdot 60 \text{ seedlings} + 6.5^{10} \text{ taproots}}$

(c) What is the long-term ratio of seedlings to taproots? Express as the simplest whole-number ratio.

The long-term ratio of seedlings to taproots is 60 to 1 because that is the ratio in the eigenvector corresponding to the largest eigenvalue.