MATA35 - Quiz 4 - Practice

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Problem 1 [4pts]. Classify the following differential equations. Is it a partial differential equation, or just an ordinary differential equation?

If it is an ODE, answer the following: what is the order? Is it linear? Is it autonomous?

(a) $y'' - (y')^2 + y = 3\ln x$ ODE, 2nd order, nonlinear, nonautonomous

(b) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = z^2$ PDE

(c) $\dot{x} = 2x - 4t$, where $\dot{x} = \frac{dx}{dt}$ ODE, 1st order, linear, nonautonomous

(d) $(y')^2 - y = 0$ ODE, 1st order, nonlinear, autonomous

Problem 2 [4pts]. Solve the following exact ODE. Make sure that in addition to finding an appropriate $f(x,y)$ such that the left hand side is the total differential, you also give your answer as an implicit equation of x and y.

$$dx(ye^{xy} + 8xy) + dy(xe^{xy} + 4x^2 - 3) = 0$$

Test for exactness:

(not strictly necessary but useful in case not exact)

$$\frac{d}{dy}\left[ye^{xy} + 8xy\right] = xye^{xy} + e^{xy} + 8x$$

$$\frac{d}{dx}\left[xe^{xy} + 4x^2 - 3\right] = xye^{xy} + e^{xy} + 8x$$

exact

$$\int\left[ye^{xy} + 8xy\right]dx = e^{xy} + 4x^2y + F(y)$$

$$\int\left[xe^{xy} + 4x^2 - 3\right]dy = e^{xy} + 4x^2y - 3y + G(x)$$

$$\Rightarrow \boxed{e^{xy} + 4x^2y - 3y = C}$$

Problem 3 [4pts]. It is straight-forward to solve first-order ODEs in two cases: (1) when they are separable, or (2) when they are exact differentials. In each of the following cases, find an appropriate substitution or integrating factor to make the equation either separable or exact.

You do NOT need to solve the equations. You may state any change of variables as multiple substitutions. Integrating factors should be simplified as much as possible.

A few hints for Quiz 4 follow (don't rely on them too much since you won't get them for the final):

Integrating factor for linear 1st order ODE:

- Given $y' + p(x)y = q(x)$
- $I(x) = \exp(\int p(x)dx)$
- If you multiple both sides by $I(x)dx$, and integrate both sides, you get $I(x)y = \int q(x)I(x)dx$, and can solve for y.

Common substitution guesses:

- $P(x,y)dx + Q(x,y)dy = 0$, where $P(tx,ty) = t^n P(x,y)$ and $Q(tx,ty) = t^n Q(x,y)$ for some integer n. Let $u = x/y$. Will get separable ODE.
- $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$.
  - If the two lines are intersecting, then let $u = a_1x + b_1y + c_1$, and $v = a_2x + b_2y + c_2$. Will get case above, and need another substitution $z = u/v$ to get to separable.
  - If the two lines are parallel, then let $u = a_1x + b_1y + c_1$. Will get separable ODE.
- Bernoulli ODE: $y' + P(x)y = Q(x)y^n$. Multiply by $(1-n)y^{-n}$. Then let $u = y^{1-n}$. Will get 1st-order linear ODE. Then need to use Integrating Factor to make exact.

(a) $y' - 4x^3y = \exp(x^4)$

1st order linear, so let Integrating Factor

$$I(x) = e^{\int -4x^3\,dx} = e^{-x^4}$$

Then we will have an exact differential.

(b) $(2x - y - 10)dx + (x - 2y + 1)dy = 0$

Lines are intersecting.

So let $u = 2x - y - 10$, and $v = x - 2y + 1$

Then will need another substitution $z = u/v$,

giving us a separable equation

Problem 4 [4pts]. Let $y' = y^2(y+1)(y+2)$. Find the equilibria of this autonomous system. Determine whether those equilibria are asymptotically stable, unstable, or semi-stable.

Equilibria at $y' = 0$, so $y = -2, -1, 0$.

-2 $\quad$ -1 $\quad$ 0

$y' = (+)(-)(-) = (+)$

$y' = (+)(-)(+) = (-)$

$y' = (+)(+)(+) = (+)$

$y' = (+)(+)(+) = (+)$

$y = -2$ is stable

$y = -1$ is unstable

$y = 0$ is semistable

Problem 5 [4pts]. Let $y' = f(x,y) = x - y^2$. Suppose you have initial conditions $y(0) = 1$. Use Euler's method to approximate the value at $y(3)$ with a step size of $\Delta x = 1$.

$x_0 = 0, \quad y_0 = 1 \qquad f(x_0, y_0) = f(0,1) = -1$

$x_1 = x_0 + \Delta x = 1, \quad y_1 = y_0 + \Delta x\, f(x_0, y_0) = 1 + 1 \cdot (-1) = 0$

$f(x_1, y_1) = f(1, 0) = 1$

$x_2 = x_1 + \Delta x = 2, \quad y_2 = y_1 + \Delta x\, f(x_1, y_1) = 0 + 1 \cdot 1 = 1$

$f(x_2, y_2) = f(2, 1) = 1$

$x_3 = x_1 + \Delta x = 3, \quad y_3 = y_2 + \Delta x\, f(x_2, y_2) = 1 + 1 = 2$

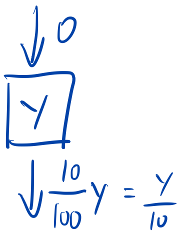
So $y(3) \approx y_3 = 2$ is the approximation via Euler's method

Problem 6 [5pts]. A tank contains 100 gallons of water. You accidentally dissolved 200 pounds of salt in that water, turning it into brine with 2 pounds salt / gallon water. You open up the drain at the bottom of the tank, which lets out 10 gallons of water per minute from the tank, and simultaneously replace it with 10 gallons of fresh water per minute.

(a) **[2pt]**. Draw this as a one-compartment model and write the corresponding ODE.

(b) **[3pt]**. How long does it take to remove half the salt from the tank? (you may find it useful to know that $\ln 2 = 0.693$ and $\ln \frac{1}{2} = -0.693$.)

(a)



$$Y' = -\frac{Y}{10}$$

(b)

$$\frac{dy}{dx} = -\frac{y}{10}$$

$$\int \frac{dy}{y} = \int -\frac{dx}{10}$$

$$\ln|y| = -\frac{x}{10} + C$$

$$y = Ce^{-\frac{x}{10}} \leftarrow \text{gen sol}$$

$$y(0) = 200$$

$$200 = Ce^{0} = C$$

$$y = 200e^{-\frac{x}{10}} \leftarrow \text{sol to IVP}$$

Want to solve for $y(x) = 100$

Solve for x when $y = 100$.

$$100 = 200e^{-\frac{x}{10}}$$

$$\frac{1}{2} = e^{-\frac{x}{10}}$$

$$\ln\frac{1}{2} = -\frac{x}{10}$$

$x = -10 \ln\frac{1}{2}$ minutes

$x = 10 \ln 2$ minutes

$x = 6.93$ minutes

any of these answers is acceptable & correct, but you must remember to specify time in minutes