MATA35 - Quiz 5 - Practice

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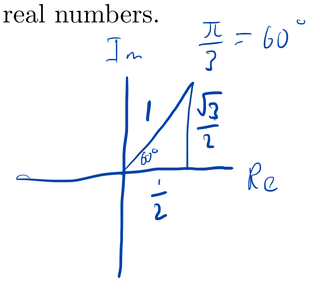
Problem 1 [6pts].

(a) Rewrite the complex expression $e^{1+\frac{\pi}{3}i}$ in the standard form $a + bi$, where a and b are real numbers.

$$e^{1+\frac{\pi}{3}i} = e^{1} e^{\frac{\pi}{3}i} = e\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$= e\left[\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right] = \frac{1}{2}e + \frac{e\sqrt{3}}{2}i$$

$\frac{\pi}{3} = 60^\circ$

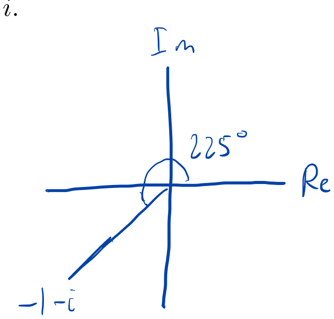


(b) Find the modulus and argument (angle) for the following complex number: $-1 - i$.

$$|-1-i| = \sqrt{2}$$

$$\operatorname{Arg}(-1-i) = 225^\circ = \frac{5}{4}\pi = -\frac{3}{4}\pi$$

all valid solutions



(c) Find the modulus and argument for $(-1-i)^{1000}$. Simplify as much as is reasonable without a calculator; e.g. you do not need to evaluate things like 2^{500}, but should simplify $\sqrt{2}^{1000} = 2^{500}$.

$$\left|(-1-i)^{1000}\right| = |-1-i|^{1000} = \sqrt{2}^{1000} = \boxed{2^{500}}$$

not simple

$$\operatorname{Arg}\left((-1-i)^{1000}\right) = 1000\cdot\operatorname{Arg}(-1-i) = 1000\cdot\frac{5}{4}\pi = 1250\pi = \boxed{0}$$

To compute, find the remainder when dividing by 2π

This simplification is because $2\pi = 360^\circ$ so we want a final answer that is between -2π & 2π

Problem 2 [5pts]. Consider the following ODE:

$$y''' + y'' - y' - y = x + 1$$

(a) Find the real homogeneous solution.

(b) Find a real particular solution

(c) Find the real general solution

(a) $\lambda^3 + \lambda^2 - \lambda - 1 = 0$

$(\lambda+1)(\lambda^2-1) = 0$

$(\lambda+1)^2(\lambda-1) = 0$

$\lambda = \begin{cases} -1, & \text{multiplicity } 2 \\ 1 \end{cases}$

$y_{hom} = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x}$

note that you should expect repeated roots, complex roots, etc., to be fair game.

(b) Ansatz for particular sol.

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$y_p''' = 0$

$y''' + y'' - y' - y = -A - Ax - B = x + 1$

$\left.\begin{aligned} -A - B &= 1 \\ -A &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= -1 \\ B &= 0 \end{aligned}$

$y_p = -x$

(c) $y_{gen} = y_h + y_p = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x} - x$

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Problem 3 [5pts]. Solve the following initial value problem:

$$\dot{x} = x + 2y$$
$$\dot{y} = 2x + 4y$$

where $x(0) = 5$ and $y(0) = 0$.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Sol 1: Reduction

$$\ddot{x} = \dot{x} + 2\dot{y}$$

$$\ddot{x} = \dot{x} + 2(2x + 4y) = \dot{x} + 4x + 8y$$

$$y = \frac{1}{2}(\dot{x} - x)$$

$$\ddot{x} = \dot{x} + 4x + 4(\dot{x} - x)$$

$$\ddot{x} = 5\dot{x}$$

$$\ddot{x} - 5\dot{x} = 0$$

Char eq: $\lambda^2 - 5\lambda = 0$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5$$

$$x = c_1 e^{0t} + c_2 e^{5t} = c_1 + c_2 e^{5t}$$

$$\dot{x} = 5c_2 e^{5t}$$

$$y = \frac{1}{2}(\dot{x} - x) = \frac{1}{2}\left(5c_2 e^{5t} - (c_1 + c_2 e^{5t})\right)$$

$$= 2c_2 e^{5t} - \frac{1}{2} c_1$$

Plugging in $\left.\begin{aligned} x(0) &= 5 = c_1 + c_2 \\ y(0) &= 0 = 2c_2 - \frac{1}{2}c_1 \end{aligned}\right\} \begin{aligned} c_1 &= 4 \\ c_2 &= 1 \end{aligned}$

$$\boxed{\begin{aligned} x &= 4 + e^{5t} \\ y &= -2 + 2e^{5t} \end{aligned}}$$

Sol 2: Matrix

$$\begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5.$$

$\lambda_1 = 0$ $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x + 2y = 0$, $\quad x = -2y$

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 5$ $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$

$x + 2y = 5x$ $\quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y = 2x$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$$

Plugging in $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$

$$c_1 = -2c_2$$

$$\Rightarrow c_1 = -2, \quad c_2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 + e^{5t} \\ -2 + 2e^{5t} \end{bmatrix}$$

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Problem 4 [4pts]. Let $\dot{z} = Az$ for each of the following 2x2 matrices A. Classify the equilibrium at the origin by type and stability.

(a) $\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda-1 & 4 \\ -1 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda + 1 + 4 = 0$$
$$(\lambda-1)^2 = -4$$
$$\lambda - 1 = \pm 2i$$
$$\lambda = 1 \pm 2i$$

$\boxed{\text{spiral, unstable}}$

(b) $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

$$\lambda = -3, -3$$

$\boxed{\text{node, asymptotically stable}}$

(c) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

$$\begin{vmatrix} \lambda+1 & -2 \\ -2 & \lambda+1 \end{vmatrix} = (\lambda+1)^2 - 4 = 0$$
$$\lambda + 1 = \pm 2$$
$$\lambda = -3, 1$$

$\boxed{\text{Saddle pt, unstable}}$

(d) $\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

$$\begin{vmatrix} \lambda & -4 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 4 = 0$$
$$\lambda^2 = -4$$
$$\lambda = \pm 2i$$

$\boxed{\text{Center stable}}$

Note: stable + asymp stable are not the same thing

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Problem 5 [5pts]. Tank A contains 100 gallons of water. Tank B contains 200 gallons of water with 50 lb of salt dissolved. Water is pumped from tank A to tank B at a rate of 10 gallons per minute. Water is pumped from tank B to tank A at a rate of 5 gallons per minute. Water is drained from tank B to the outside at a rate of 5 gallons per minute. Pure water is added to tank A at a rate of 5 gallons per minute.

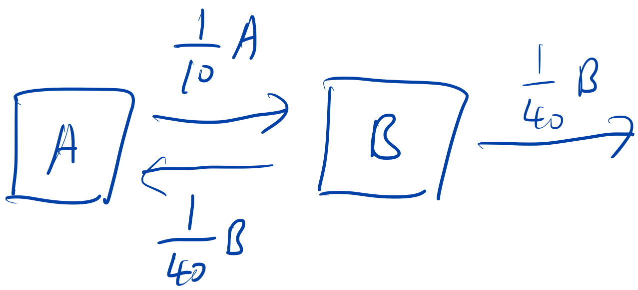
Let $A(t)$ and $B(t)$ denote the total amount of salt present in tanks A and B respectively.

(a) Draw a 2-compartment model for A and B.

(b) Write a system of two first-order differential equations modelling the system.

(c) Find the equilibrium values for A and B.

(a)



(b)
$$\dot{A} = -\frac{A}{10} + \frac{B}{40}$$
$$\dot{B} = \frac{A}{10} - \frac{B}{20}$$

(c)
$$\begin{cases} 0 = -\frac{A}{10} + \frac{B}{40} \\ 0 = \frac{A}{10} - \frac{B}{20} \end{cases} \Rightarrow \begin{cases} 0 = -4A + B \\ 0 = 2A - B \end{cases} \Rightarrow B = 2A$$

$$-2A = 0$$

$$\boxed{\begin{cases} A = 0 \\ B = 0 \end{cases}}$$

only equilibrium

Formulas that may be useful:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{cosx}$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x \qquad (\arctan x)' = \frac{1}{1+x^2}$$

$$\sin^2 x + \cos^2 x = 1 \qquad \sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

Integration by parts: $\int u\ dv = uv - \int vdu$

Common trignometric values table

θ in Degrees	θ in Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
180	π	0	-1	0
270	$\frac{3\pi}{2}$	-1	0	undefined
360	2π	0	1	0