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MATB44 - 2019

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Differential Equations I

Welcome

Def. A classical ordinary differential equation (ODEs) is a functional relationship of the form

$$F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$$

for the unknown function $x \in C^k(J)$, $J \subseteq \mathbb{R}$, and its derivatives $x^{(j)}(t) = \dfrac{d^j x(t)}{dt^j}$, $j \in \mathbb{N}_0$,

where $F \in C(U)$ with U an open subset of $\mathbb{R}^{k+2}$.

Def. A solution to an ODE is a function $\phi \in C^k(I)$, $I \subseteq J$ an interval, s.t.

$$F(t, \phi(t), \phi^{(1)}(t), \dots, \phi^{(k)}(t)) = 0.$$

Set: Collection of items

Ex $\{1, \text{apple}, ☆\}$, $\mathbb{N}_0 = \{0, 1, 2, 3 \dots\}$

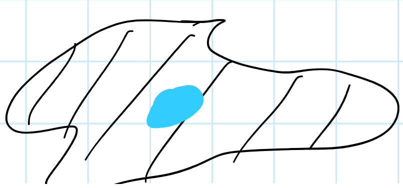
$\mathbb{R}$ = all real numbers, e.g. $1, \pi, 2/7, 5.43, 42$

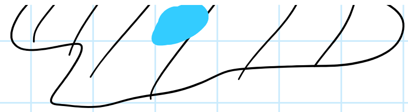
$\mathbb{R}^n = (x_1, x_2, \dots, x_n)$ - n-dim. space

Subset: A set contained in another set

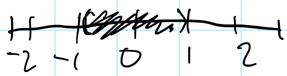
$\{☆\} \subseteq \{1, \text{apple}, ☆\}$ $\quad ☆ \in \{1, \text{apple}, ☆\}$

positive even numbers $\subseteq \mathbb{N}_0$

 $\subseteq \mathbb{R}^2$

 $\subseteq \mathbb{R}$

Open Subset: A set that doesn't contain boundary pts.



$$(-1,1) \subseteq \mathbb{R}$$

does not contain -1 or 1.

More on Banach spaces and limits later.

$C^k(J)$: k-times differentiable functions on $J \subseteq \mathbb{R}$

$C^0(J)$: continuous functions

$C^1(J)$: functions with continuous 1st derivative

ex $f(x) = 0 \in C^k(\mathbb{R}) \quad \forall k \in \mathbb{N}_0 \quad (C^\infty(\mathbb{R}))$

(Leibniz notation) $\frac{df(x)}{dx} = 0$

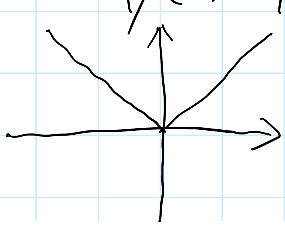
(Lagrange notation)

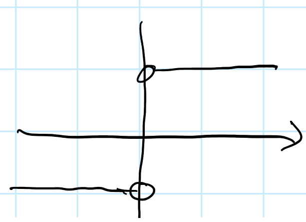
$$\left.\begin{aligned} f(x) &= x^2 + 3x - 1 \in \\ f'(x) &= 2x + 3 \\ f''(x) &= 2 \\ f'''(x) &= 0 \end{aligned}\right\} \in C^\infty(\mathbb{R})$$

(Newton notation)

$$\left.\begin{aligned} x(t) &= \sin(t) \\ \dot{x}(t) &= \cos(t) \\ \ddot{x}(t) &= -\sin(t) \\ \dddot{x}(t) &= -\cos(t) \\ \overset{(4)}{x}(t) &= \sin(t) \end{aligned}\right\} \in C^\infty(\mathbb{R})$$

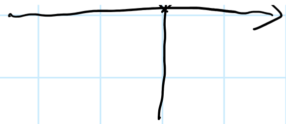
$y(x) = |x| \in C^0(\mathbb{R}) \quad \notin C^1(\mathbb{R})$

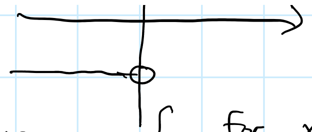




Look on $(0,1) \subset \mathbb{R}$

$\in C^\infty(0,1)$





$\in C^{\infty}(0,1)$

$y(x) = |x|$ $\quad y'(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$

Functional relationship: implicit function

$F(x, y) = 0$

ex. $x + y = 0$

$\sin(x) - y^2 = 0$

$F(t, x, x') = 0$

ex. $t^2 - x + x' = 0$

$x - x' = 0$

$t + |x'| \sin(x) = 0$

Solution: A function that can be plugged into an ODE.

Ordinary equations like $x + 3 = -1$ have solutions like $x = -4$

ODEs like $\ddot{x} - \dot{x} + 2t - 2 = 0$ have solutions like $x = t^2$

$x = t^2$

$\dot{x} = 2t \qquad 2 - 2t + 2t - 2 = 0$

$\ddot{x} = 2$

Try $x = 2t^2 \qquad 4 - 4t + 2t - 2 \stackrel{?}{=} 0$ ✗

$\dot{x} = 4t$

$\ddot{x} = 4$

Try $x = e^t t^2$

$\dot{x} = e^t \cdot 2t + e^t t^2$

$\ddot{x} = 2e^t + e^t \cdot 2t + e^t \cdot 2t + e^t t^2$ ✗

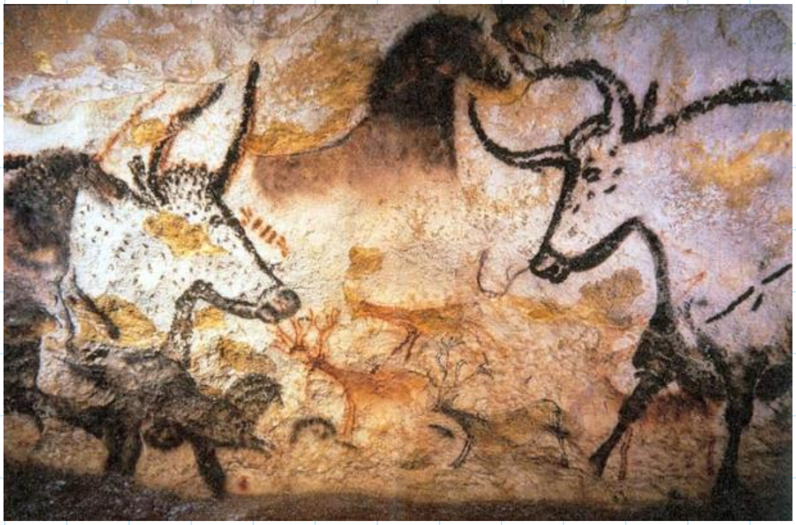
$2e^t + \underline{4te^t} + e^t t^2 - (\underline{e^t \cdot 2t} + e^t t^2) + 2t - 2 \stackrel{?}{=} 0$

Soln. $x = t^2 + C_1e^t + C_2$

Exercise for the audience

Radiocarbon dating: Living organisms contain carbon. There are two different isotopes C^{12} and C^{14}. C^{12} is stable, but C^{14} is not. After a tree dies, C^{14} is no longer replaced & slowly decays away.

In 1940, cave paintings were discovered in Lascaux, France.



The charcoal used came from dead trees. Chemists were able to determine that 85.5 % of the C^{14} present at death had decayed. When did the trees die / when were the paintings painted?

Hint: every atom of C^{14} has the same chance of decaying over a specific time period.

Half-life of C^{14} is 5730 years.

Let t = time since death of a tree

$x(t)$ = amount of C^{14} in the tree

The decay rate is proportional to the amount.

$$\dot{x}(t) = -k\,x(t)$$

$$\frac{dx}{dt} = -kx$$

$$\int \frac{1}{x}\frac{dx}{dt}\,dt = \int -k\,dt$$

$$\int \frac{1}{x}\,dx = -kt + C_2$$

$$\ln(x) + C_1 = -kt + C_2$$

$$\ln(x) = -kt + C_3$$

$$x = e^{-kt + C_3} = e^{C_3} e^{-kt} = C_4 e^{-kt}$$

$$x(5730) = 0.5\,x(0)$$

$$Ce^{-5730k} = 0.5\,Ce^{-k\cdot 0} = 0.5C$$

$$e^{-5730k} = 0.5$$

$$-5730k = \ln 0.5$$

$$k = 0.000121$$

$$x = Ce^{-0.000121k}$$, where C is the starting amount.

Started with A_0 C^{14}.

End with $(1 - 0.855)A_0$ of C^{14}

$$0.145\,A_0 = A_0\,e^{-0.000121k}$$

$$e^{-0.000121k} = 0.145$$

$$\Rightarrow t \approx 15{,}959 \text{ years}$$

Classification of ODEs:

ODE: $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

t = independent variable (often time)

↳ If multiple ind. var. $t_1, t_2, \dots$ then these are partial differential equations

x = dependent variable (often space)

↳ x is a function of t i.e. $x(t)$

↳ multiple dependent variables $\Rightarrow$ system of ODEs

Def. The order of an ODE is the highest derivative to appear in F.

e.g. $t + x^2 + \sin x'' = 0 \Rightarrow$ 2nd order

$\dot{x} = -kx \Rightarrow$ 1st order

$\frac{d^5 x}{dt^5} - \frac{d^2 x}{dt^2} = 0 \Rightarrow$ 5th order

$y'' - y'' + y' + y = 0 \Rightarrow$ 1st order

Note: Implicit functions are hard to work with, so we normally a more explicit form.

$$x^{(k)} = f(t, x, x^{(1)}, \dots, x^{(k-1)})$$

With this form, we can easily write **systems of ODEs** where $x(t) : \mathbb{R} \to \mathbb{R}^n$ is a vector valued function.

(or equivalently, we have multiple dependent variables $x_1(t) : \mathbb{R} \to \mathbb{R}$, $x_2(t) : \mathbb{R} \to \mathbb{R}$, etc.)

$$x_1^{(k)} = f_1(t, x, x^{(1)}, \dots, x^{(k-1)})$$

$$\vdots \qquad\qquad \swarrow (x_1^{(k-1)}, \dots, x_n^{(k-1)})$$

$$x_n^{(k)} = f_n(t, x, x^{(1)}, \dots, x^{(k-1)})$$

↳ $(x_1, x_2, \dots, x_n)$

The system is **linear** if we can rewrite it to separate out the $x^{(j)}$'s as follows:

The system is linear if we can rewrite it to separate out the $x_\ell^{(j)}$'s as follows:

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=0}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}.$$

Ex. $\ddot{x} = t + \sin(t)x + t^5 \dot{x}$ ~ linear

$\begin{cases} x_1'' = x_2' \\ x_2'' = t + e^t \cos(t) x_1' + x_2' - x \end{cases}$ – linear

$\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt} \cdot x$ – nonlinear

$x'' = xt$ – linear

A linear system of ODE's is homogeneous if $g_i(t) \equiv 0$.

Ex. $\ddot{x} = \underline{t} + \sin(t)x + t^5\dot{x}$ – nonhomogeneous

$\left.\begin{cases} x_1'' = x_2' \\ x_2'' = \underline{t} + e^t \cos(t) x_1' + x_2' - x \end{cases}\right\}$ nonhomogeneous

$x'' = xt$ – homogeneous

Note that any system of ODEs can be transformed into a new first-order system by changing the dependent variables to $Y = (x, x^{(1)}, \ldots, x^{(k-1)})$

$$x^{(k)} = f(t, x, x^{(1)}, \ldots, x^{(k-1)}) \longrightarrow \begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_{k-1} &= y_k \\ \dot{y}_k &= f(t,y) = f(t, y_1, y_2, \ldots, y_k) \end{aligned}$$

Ex. $\ddot{x} + t^2 \cos(\dot{x}) = tx$

Ex. $\ddot{x} + t^2 \cos(\dot{x}) = tx$

Let $y_1 = x, \quad y_2 = \dot{x}$

$\dot{y}_1 = y_2$

$\dot{y}_2 = \ddot{x} = tx - t^2\cos(\dot{x}) = ty_1 - t^2\cos(y_2)$

If there is no direct dependence on t, the system is autonomous.

Ex $\dot{x} = -kx$ (radiocarbon dating) - autonomous

Often known as tim-invariant systems.

Note that we can tranform a nonautonomous system to an autonomous systen using the same trick, by including t as a dependent variable.

Ex. $xx' + tx'' = 1$ - 2nd order

$x' + x = 0$ - 1st order, linear, homogeneous, autonomous

$x'' + tx = 0$ - 2nd order, linear, homogeneous

$x' - 3x = 5$ - 1st order, autonomous, linear

$x'' = tx' - 3t$ - 2nd order, linear

$x'(t) + x(t) = 0$, dependence on t is indirect so autonomous