2019-Sep-05

Wednesday, September 4, 2019 5:10 PM

Outline:

- Solve first-order autonomous equation (in general)
- Give a couple practice examples

Consider a first-order **autonomous** equation

$$\dot{x} = f(x), \qquad f \in C(\mathbb{R})$$

Remember that I also called these **time-invariant** equations.
The reason is because if $\phi(t)$ is a solution, then so is $\psi(t) = \phi(t-t_0)$. (HW 1.8)

Thus, we only have to consider solutions starting at time $t=0$.
Let's try to find the solution that starts at a particular point x_0.
i.e $x_0 = x(0)$.

If $f(x_0) \neq 0$, we can divide to get

$$\frac{\dot{x}}{f(x)} = 1$$

$$\frac{\dot{x}(t)}{f(x(t))} = 1 \quad \text{write out the } t\text{-dependence explicitly}$$

$$\frac{\dot{x}(s)}{f(x(s))} = 1 \quad \text{change } t \to s$$

$$\int_0^t \frac{\dot{x}(s)}{f(x(s))}\,ds = \int_0^t ds = t$$

Let's change variables setting $y = x(s)$.
Then $\displaystyle\int_{x(0)}^{x(t)} \frac{dy}{f(y)} = t.$

Heuristically, (or more formally, using differential forms)

$$\frac{1}{f(x)}\frac{dx}{dt} = 1$$

$$\frac{dx}{f(x)} = dt$$

$$\frac{dy}{f(y)} = ds \quad \text{change } x \to y,\ t \to s$$

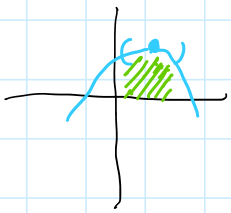
$$\int_{y=x(0)}^{y=x(t)} \frac{dy}{f(y)} = \int_{s=0}^{s=t} dt$$

$$\int_{x(0)}^{x(t)} \frac{dy}{f(y)} = t$$

Let $\displaystyle F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}$. Then clearly $\displaystyle F(x(t)) = \int_{x_0}^{x(t)} \frac{dy}{f(y)} = t.$

Lemma $F(x)$ is strictly monotone near x_0.
↳ always going up, or always going down.

proof. $f(x_0) \neq 0$, and $f \in C(\mathbb{R})$.
Two cases: $f(x_0) > 0$ or $f(x_0) < 0$.
We focus on $f(x_0) > 0$.
Then by continuity, $\exists\, \delta > 0$ s.t. $\forall x \in (x_0 - \delta, x_0 + \delta)$, $f(x) > 0$.



$$f(x) > 0 \implies \frac{1}{f(x)} > 0.$$

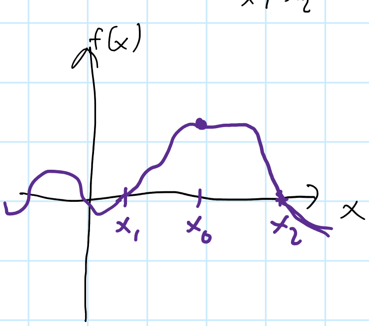
Thus $F(x)$ on $(x_0 - \delta, x_0 + \delta)$ is strictly monotone positive.
The converse holds true for $f(x_0) < 0$. $\square$

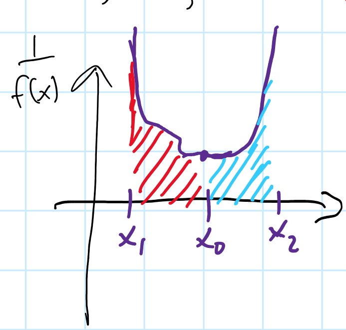
Since $F(x)$ is strictly monotone near x_0, it is invertible.
We can then obtain a unique solution to our ODE
$$\phi(t) = F^{-1}(t), \qquad \phi(0) = F^{-1}(0) = x_0.$$

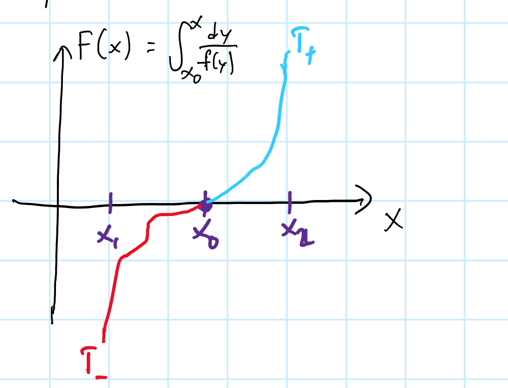
We only showed $\phi(t)$ to be defined in a small δ-ball around x_0.
How far can we define $\phi(t)$ by $F^{-1}(t)$?

Let (x_1, x_2) be a maximal interval around x_0 where $f(x)$ is positive.
(we consider only the positive case, but the negative is analogous)

Define $T_+ = \lim_{x \uparrow x_2} F(x) \in [0, \infty]$, $\quad T_- = \lim_{x \downarrow x_1} F(x) \in [-\infty, 0)$







Note that $F(x)$ has a nonzero continuous derivative on (x_1, x_2).
Then by the inverse function theorem, $F^{-1}(x)$ has a nonzero
continuous derivative on $(F(x_1), F(x_2)) = (T_-, T_+)$

Then by the inverse function theorem, $F^{-1}(x)$ has a nonzero continuous derivative on $(F(x_1), F(x_2)) = (T_-, T_+)$.

So $\phi \in C^1((T_-, T_+))$ and $\lim_{t \uparrow T_+} \phi(t) = x_2$, $\lim_{t \downarrow T_-} \phi(t) = x_1$.

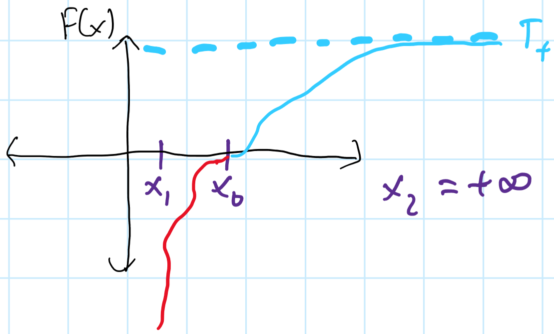
Note that ϕ is defined for all $t > 0$ iff $T_+ = \int_{x_0}^{x_2} \frac{dy}{f(y)} = +\infty$.

i.e. if $\frac{1}{f(x)}$ is not integrable near x_2.

(similarly for $t < 0$ and integrability near x_1)

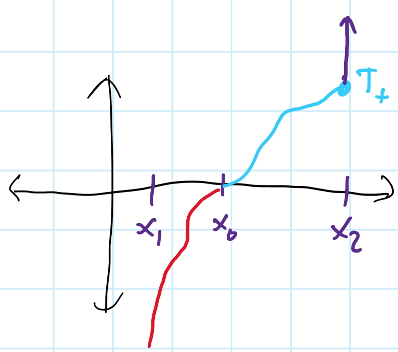
If $T_+ < +\infty$, two cases:

Case 1: $x_2 = +\infty$



When taking the inverse, x_2 $\lim_{t \uparrow T_+} \phi(t) = \infty$, so cannot extend solution continuously past T_+

Case 2: $x_2 < +\infty$



Recall (x_1, x_2) is a maximal interval on which $f(x) > 0$. Thus $f(x_2) = 0$. Then we can extend ϕ by setting $\phi(t) = x_2$ for $t \geq T_+$

Examples

$\dot{x} = x$, with $x_0 = x(0) > 0$. (Recall radiocarbon dating)

Last time, we did

$x' = x$, with $x_0 = x(0) > 0$. (Recall radiocarbon dating)

Last time, we did

$$\frac{dx}{dt} = x \Rightarrow \int \frac{dx}{x} = \int dt \Rightarrow \ln x = t + C_1$$

$$\Rightarrow x = C_2 e^t \qquad (C_2 = e^{C_1})$$

$$\Rightarrow x = x_0 e^t.$$

This time, more rigorously:

Then $f(x) = x$.

And a maximal nonzero interval is $(x_1, x_2) = (0, \infty)$.

$$F(x) = \int_{x_0}^{x} \frac{dy}{y} = \ln y \Big|_{y=x_0}^{y=x} = \ln x - \ln x_0 = \ln \frac{x}{x_0}$$

$$T_+ = \lim_{x \uparrow x_2} F(x) = \ln \frac{\infty}{x_0} \longrightarrow \infty,$$ which implies ϕ is defined for all $t > 0$.

$$\phi(t) = F^{-1}(t).$$

$$t = F\left(F^{-1}(t)\right) = \ln \frac{F^{-1}(t)}{x_0}$$

$$\Rightarrow F^{-1}(t) = x_0 e^t$$

$$\phi(t) = x_0 e^t.$$

$\square$