Problem Set 2 - Due Oct 7, 11:59pm

[Your name] and [student ID]
MATB44H3-2019

September 24, 2019

Problem 1. Consider the autonomous equation

$$\frac{dx}{dt} = -\left(1 - \frac{x}{T}\right)\left(1 - \frac{x}{K}\right)x,$$

for constants $T, K \in \mathbb{R}$ and $0 < T < K$. When $x > 0$, this equation can be thought to describe logistic growth with a quorum. We wish to qualitatively analyze the limiting behavior of this system as $t \to \infty$.

- Plot a graph of $\frac{dx}{dt}$ (vertical) vs. x (horizontal).
- Draw the direction field along the x axis, with arrows pointing the direction the system will evolve.
- The zeros of $\frac{dx}{dt}$ are called its **critical points**. How many critical points are there? What are they, and which ones are stable?

Solution. Collaborators: Donald Duck and Mickey Mouse ☐

Problem 2. Find the first 3 Picard iterations after $x_0(t)$ of the particular solution of each of the following equations:

(i) $\dot{x} = t + x$, where $x(0) = 1$

(ii) $\dot{x} = e^t + x$, where $x(0) = 0$

Solution. Collaborators: Gandalf Greyhame and Bilbo Baggins ☐

Problem 3. Given a function $x : \mathbb{R} \to \mathbb{R}$, prove that the map

$$||x|| \overset{\text{def}}{=} \sup_{t \in [0,1]} |x(t)|$$

is a norm.

Solution. Collaborators: Mark Watney ☐

Problem 4. Recall Newton's method for finding the zeros of a twice continuously differentiable function $f(x)$, given by

$$x_{n+1} = K(x_n), K(x) = x - \frac{f(x)}{f'(x)}.$$

That is to say, we start from a point x_0, and then repeatedly apply $K(x)$ to approximate a nearby 0 of $f(x)$ through the sequence $x_0, x_1, \ldots$.

Prove that Newton's method works:

- First show that if $f(\bar{x}) = 0$ and $f'(\bar{x}) \neq 0$, then $\bar{x}$ is a fixed point of K.
- Then show that there is a corresponding closed interval C around $\bar{x}$ such that we can apply the Banach fixed point theorem to prove uniqueness of the solution in that interval.

Solution. Collaborators: No one. ☐