1.4b Higher order equations

Wednesday, January 13, 2021 2:38 AM

Let's review linear ODEs:

$$a_k(t)\frac{d^k}{dt^k}x+\cdots+a_1(t)\frac{dx}{dt}+a_0(t)x=b(t)$$

$$a_k(t)\frac{d^k}{dt^k}x(t)+\cdots+a_1(t)\frac{dx(t)}{dt}+a_0(t)x(t)=b(t)$$

(1) Find k linearly ind. sols to homogeneous

$$a_k(t)\frac{d^kx}{dt^k}+\cdots+a_1(t)\frac{dx}{dt}+a_0(t)x=0$$

Called $x_1(t), x_2(t), \dots, x_k(t)$

x_1+x_2 (x_1 and x_2 are sols)

$$a_k(t)\frac{d^k}{dt^k}(x_1+x_2)+\cdots+a_1(t)\frac{d}{dt}(x_1+x_2)+a_0(t)(x_1+x_2)=0$$

$$\Leftarrow \left(a_k(t)\frac{d^k}{dt^k}x_1+\cdots+a_1(t)\frac{d}{dt}x_1+a_0(t)x_1\right)+\left(a_k(t)\frac{d^k}{dt^k}x_2+\cdots+a_1(t)\frac{d}{dt}x_2+a_0(t)x_2\right)=0$$

$x_h(t)=\sum_{i=1}^{k}c_ix_i(t)$ $\forall c_i\in\mathbb{R}$ is a sol. to hom.

- If $a_i(t)\equiv a_i$ constants, then we can use the characteristic eqn

$$a_k\lambda^k+\cdots+a_1\lambda+a_0=0.$$

Then $t^\beta e^{\lambda t}$ is a sol. to hom. eqn, $\beta=\{0,\dots,m(\lambda)-1\}$ $m(\lambda)$ = multiplicity of λ as a solution to char. eqn.

$$a_kD^kx+\cdots+a_1Dx+a_0x=0$$

$$(a_kD^k+\cdots+a_1D+a_0)x=0$$

Suppose λ is a root of multiplicity $m(\lambda)$

$$\Rightarrow (\sim\sim\sim)\underbrace{(D-\lambda)^{m(\lambda)}x}_{\text{Want this to be 0.}}=0$$

Linear difference equations

$$a_k(t)x(t+k)+\cdots+a_1(t)x(t+1)+a_0(t)x(t)=b(t)$$

$$\cancel{a_k(t)x_{t+k}+\cdots+a_1(t)x_{t+1}+a_0(t)x_t=b_t}$$

(1) Find k lin. ind. sol. to homogeneous

$$a_k(t)x(t+k)+\cdots+a_1(t)x(t+1)+a_0(t)x(t)=0$$

called $x_1, x_2, \dots, x_k$.

cx_1

$$a_k(t)\,cx_1(t+k)+\cdots+a_1(t)\,cx_1(t+1)+a_0(t)\,cx_1(t)$$

$$=c\left(a_k(t)x_1(t+k)+\cdots+a_1(t)x_1(t+1)+a_0(t)x_1(t)\right)=0$$

$x_h=\sum_{i=1}^{k}c_ix_i$ $\forall c_i\in\mathbb{R}$ is a sol. to hom. eqn.

- If $a_i(t)\equiv a_i$ constants, then we can use the characteristic eqn

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Ex. 1.7 $x(t+3)+x(t+2)+x(t+1)+x(t)=0$

Char. eq. $\lambda^3+\lambda^2+\lambda+1=0$

$$(\lambda+1)(\lambda^2+1)=0$$

$$\Rightarrow \lambda=-1,\ i,\ -i.$$

ind. sol. -1^t, i^t, $(-i)^t$

Want this to be 0.

$$(D-\lambda)^{m(\lambda)} x = 0$$

Exp. Shift Thm (Thm 24.56 in Tenenbaum Pollard)

If $x = ue^{\lambda t}$, then

$$(D-\lambda)^{m(\lambda)} ue^{\lambda t} = e^{\lambda t} D^{m(\lambda)} u$$

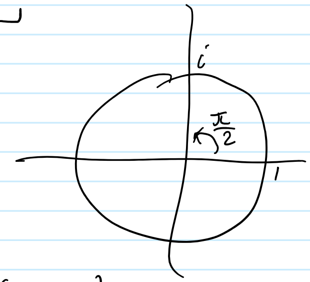
Need $D^{m(\lambda)} u = 0$

$$u = c_{m(\lambda)-1} t^{m(\lambda)-1} + c_{m(\lambda)-2} t^{m(\lambda)-2} + \cdots + c_1 t + c_0$$

ind. sol. $\underbrace{-1^t}, \underbrace{i^t}, \underbrace{(-i)^t}$

$$i = e^{\frac{\pi}{2} i}, \quad -i = e^{-\frac{\pi}{2} i}$$

$$i^t = e^{\frac{\pi}{2} i t}, \quad (-i)^t = e^{-\frac{\pi}{2} i t}$$



$$e^{\frac{\pi}{2} i t} = \cos\left(\frac{\pi}{2} t\right) + i \sin\left(\frac{\pi}{2} t\right)$$

$$e^{-\frac{\pi}{2} i t} = \cos\left(\frac{\pi}{2} t\right) - i \sin\left(\frac{\pi}{2} t\right)$$

$\Rightarrow$ another two lin. ind. sol $\cos\left(\frac{\pi}{2} t\right)$, $\sin\left(\frac{\pi}{2} t\right)$

$$\Rightarrow x_h(t) = c_1 (-1)^t + c_2 \cos\left(\frac{\pi}{2} t\right) + c_3 \sin\left(\frac{\pi}{2} t\right)$$

pg. 16 of Allen's textbook, Ca 3: Suppose $\lambda_{1,2} = A \pm iB = r(\cos\phi + i \sin\phi)$

where $r = \sqrt{A^2+B^2}$, $\phi = \arctan\left(\frac{B}{A}\right)$.

$$c_1 r^t \cos(t\phi) + c_2 r^t \sin(t\phi)$$

(2) Find any sol. $x_p(t)$ to inhom. eqn.

- Method of undetermined coeff.
- Variation of parameters

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$$x(t+3) + x(t+2) + x(t+1) + x(t) = 4t$$

Guess $x_p = k_1 t + k_2$. Plug it in:

$$k_1(t+3) + k_2 + k_1(t+2) + k_2 + k_1(t+1) + k_2 + k_1 t + k_2 = 4t$$

$$\Rightarrow \underbrace{4k_1 t} + 6k_1 + 4k_2 = \underbrace{4t}$$

$4k_1 t = 4t \qquad 6k_1 + 4k_2 = 0$

$\Rightarrow k_1 = 1. \qquad \Rightarrow k_2 = -\frac{3}{2}$

$$x_p = t - \frac{3}{2}$$

(3) Gen sol. $x(t) = x_h(t) + x_p(t)$ by Principle of Superposition

(3) Gen sol $x = x_h + x_p$ by Principle of Superposition

Principle of Superposition

Principle of Superposition

$$x = x_h + x_p = c_1(-1)^t + c_2\cos\left(\frac{\pi}{2}t\right) + c_3\sin\left(\frac{\pi}{2}t\right) + t - \frac{3}{2}.$$

Def. 1.7 The zeros $\lambda_1, \dots, \lambda_k$ of the char eqn $a_k\lambda^k + \dots + a_1\lambda + a_0 = 0$ are often called eigenvalues. If for some i and for all j,

$|\lambda_i| \geq |\lambda_j|$, then λ_i is a dominant eigenvalue

or, $|\lambda_i| > |\lambda_j|$, then λ_i is the strictly dom. eigenvalue.

Asymptotically, the behavior of a typical solution is controlled by the dominant eigenvalue, e.g. if λ_i is a dominant eigenvalue, and $|\lambda_i| < 1$, then solutions converge to 0.